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## **OxCarre Research Paper 129**

# **The Elephant in the Ground: Managing Oil and Sovereign Wealth**

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**THE ELEPHANT IN THE GROUND:  
MANAGING OIL AND SOVEREIGN WEALTH**

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**Abstract**

Many oil exporters accumulate large sovereign wealth funds, though their portfolio allocation does not take into account below-ground assets, like oil. Similarly, the above-ground portfolio does not affect the decision to extract oil. This paper shows that subsoil oil wealth should change a country's above-ground asset allocation in two ways. First, the holding of all risky assets is leveraged because there is additional wealth outside the fund. Second, more (less) is invested in financial assets that are negatively (positively) correlated with oil to hedge against the riskiness of subsoil exposure. Furthermore, if marginal oil rents move pro-cyclically with the value of the financial assets in the fund, then oil will be extracted slower than predicted by the standard Hotelling rule. This leaves a buffer of oil to be extracted when both oil prices and asset returns are high. Finally, any unhedged residual volatility must be managed through additional precautionary saving.

**Keywords:** oil, portfolio allocation, sovereign wealth fund, optimal extraction

**JEL codes:** E21, F65, G11, G15, O13, Q32, Q33

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## 1. Introduction

Many countries blessed with natural resources - such as oil, natural gas, copper or diamonds (“oil” for short) - accumulate sovereign wealth funds with the proceeds from selling below-ground assets. These funds can comprise a large part of commodity exporters’ wealth. Azerbaijan’s US\$ 33 billion fund accounts for almost half of GDP, Qatar’s US\$ 115 billion fund accounts for almost two thirds of GDP, Saudi Arabia’s fund is approximately four-fifths of GDP, Norway’s fund is nearly one and a half times GDP, and the tiny island of Kiribati’s fund is double its GDP (SWF Institute, 2013; IMF, 2013). Collectively commodity sovereign wealth funds hold over US\$ 3 trillion in financial assets (SWF Institute, 2013).

Commodity sovereign wealth funds are used to smooth consumption of below-ground wealth across generations, because this wealth is temporary. They are also used to insulate government budgets from volatile commodity prices, allowing the budgetary process to be conducted with more certainty. While the funds are professionally managed and use modern portfolio theory when allocating their financial assets, it is less clear whether their investment strategy takes proper account of commodity price volatility, and the stock of subsoil reserves. It is also not clear how oil- and asset-price volatility should affect the optimal depletion of oil reserves. These are both important questions for commodity exporters, because commodity prices are notoriously volatile and below-ground assets can be worth more than the financial assets held in the fund.

Our objective is therefore to answer the following four questions regarding how below-ground assets should influence above-ground portfolios, and vice-versa. First, how should assets above the ground be allocated given the large and volatile stock of assets below the ground? Second, how should asset allocation and oil extraction affect the rate at which financial and oil wealth is consumed? Third, how does this change if financial markets are incomplete, so that oil shocks cannot be completely hedged in the portfolio? Finally, how does the above-ground allocation of financial assets affect how quickly below-ground assets are optimally extracted?

We will show that the fund should be designed to offset the exposure to subsoil oil, use precautionary savings to manage any residual volatility, and extract oil more slowly if marginal oil rents are positively correlated with the market.

We show that offsetting subsoil oil involves treating it as part of total wealth, which leads to additional demand for risky assets in the sovereign wealth fund in two ways: for leverage and for hedging. The leverage demand involves leveraging up all the holdings of risky assets by borrowing the safe asset, or going “short”, compared to the case without oil. The leverage factor is the ratio of oil wealth to financial

wealth, so these additional holdings of risky financial assets are reversed as the fund matures and oil reserves are depleted. The hedging demand involves holding more (less) of financial assets that are negatively (positively) correlated with oil price shocks, after adjusting for the correlations between these assets. Oil price volatility can be fully offset by changing the weights of financial assets if markets are complete, that is if the oil price is driven by the same underlying shocks as (is “spanned” by) the market.

Precautionary savings should be used to manage any residual volatility in the overall portfolio. The fund allows consumption to be smoothed in two ways: in expectation and in variance. In expectation, consumption will be a constant share of total wealth independent of when oil is extracted. This is reminiscent of the permanent income hypothesis and of Hartwick’s (1977) rule for replacing below-ground with above-ground assets. In variance, consumption will be insulated from oil price shocks as the fund will offset them as much as possible against other assets. The policy maker will thus only be exposed to residual volatility, which then must be managed by precautionary saving. If the oil price is not fully spanned by financial assets then there will be more residual oil price volatility, and more precautionary savings.

Finally, we show that, if marginal oil rents are positively correlated with the market, oil will be extracted more slowly. This leaves a buffer in oil reserves to be extracted when oil prices are high, giving rise to a double whammy: oil can be sold at a higher price *and* the oil rents can then be invested for a higher return. For example, both oil prices and equities perform well in anticipation of a global upturn, in which case oil extraction should be slowed down to benefit from the double whammy. If oil prices are negatively correlated with the market, oil extraction will on average be faster. This might occur during a political crisis in which oil prices typically boom and share prices fall, leading to a positive rather than a negative correlation between marginal oil rents and the marginal utility of an extra dollar in the fund.

Our analysis combines three existing strands of literature: on portfolio theory, precautionary savings and natural resource extraction. First, we employ portfolio theory to decide how to invest in risky assets. This builds on mean-variance theory to construct a diversified portfolio based on the co-movement between financial assets (Markowitz, 1952; 1959). The total amount to be invested in risky assets should be separated from the share invested in each asset (Tobin, 1958). As all investors have equal information and markets are complete, they will hold the market portfolio as used in the capital asset pricing model (Sharpe, 1964). We will use the continuous-time formulation which solves the optimal portfolio

allocation and consumption-saving decisions simultaneously (Merton, 1990) and extend it to allow for subsoil oil wealth.<sup>1</sup>

Second, we use the theory of the precautionary demand for saving. This shows that volatile income should be managed by curbing consumption in the short run to build up a buffer stock of savings.<sup>2</sup> (e.g., Leland, 1968; Sandmo, 1970; Zeldes, 1986; Kimball, 1990; Carroll and Kimball, 2008). In our context precautionary saving is used to manage any residual volatility that cannot be managed by diversifying financial assets and varying the rate of oil extraction. This extends earlier work on precautionary saving in safe assets in a sovereign wealth fund to cope with oil price volatility (Bems and de Carvalho Filho, 2011; van den Bremer and van der Ploeg, 2013).

Third, we extend the theory on the optimal rate of extraction of non-renewable resources (e.g., Hotelling, 1931; Solow, 1974; Gaudet, 1972). It has been shown that oil extraction should be more rapid if oil prices are volatile and the marginal cost of oil extraction is convex, as a form of “extractive prudence” (Pindyck, 1980, 1981). This assumes that oil wealth is only risky beneath the ground. Others have studied extraction with stochastic oil prices, growth and capital accumulation (Gaudet and Khadr, 1991; Atewamba and Gaudet, 1992). We now acknowledge that oil wealth is also risky above the ground, as it will be invested in risky financial assets. With linear marginal extraction cost we find that oil extraction slows down if the oil price and financial markets are positively correlated.

Combining these different insights and taking them further, we find that optimal behavior is described by three equations. The allocation of financial assets is described by suitably modified CAPM equations; consumption and precautionary saving by a stochastic Euler equation; and the optimal rate of oil extraction by a modified stochastic Hotelling rule. Although the issue has been identified in the empirical literature, full integration of these three strands of literature has not been discussed yet. Our results are mostly analytical, but inspired by earlier empirical results. For example, using the correlation of oil prices with other financial assets in simulations indicates that Norway’s exposure to aggregate oil price volatility can be halved if oil wealth is hedged in the sovereign wealth fund (Gintschel and Scherer, 2008) and that the fund should invest less aggressively in risky assets as it ages (Scherer, 2009; Balding and Yao, 2011). These studies focus on asset allocation but do not consider the optimal consumption-saving decisions or the optimal time path for the rate of oil extraction.

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<sup>1</sup> This is related to extensions dealing with a non-tradable stream of income in the context of university endowments (Merton, 1993; Brown and Tiu, 2012), labor income including endogenous effort (Bodie et al., 1992; Wang et al., 2013), non-tradable and uninsurable income (Svensson and Werner, 1993; Koo, 1998) and non-financial stores of wealth such as housing (Flavin and Yamashita, 2002; Sinai and Souleles, 2005; Case et al., 2005).

<sup>2</sup> This requires that individuals are “prudent”, or that the third derivative of the utility function is positive.

Norway has one of the largest and best managed commodity funds in the world.<sup>3</sup> Its Government Pension Fund Global (GPF) was established in 1990 to smooth expenditure financed from oil after a period of fiscal volatility in the 1970s and 1980s. The GPF must allocate its assets across a diversified portfolio (60% equity tracking the FTSE All Cap Index, up to 40% bond of which 70% government bonds and 30% corporate bonds, both tracking Barclays indices, and up to 5% real estate tracking the Investment Property Databank’s Global Property Benchmark) and release approximately 4% of the fund for the general budget each year. This approach is consistent with standard portfolio theory, but fails to take account of a large and volatile stock of wealth beneath the soil (the “elephant in the ground”) which matches the wealth in the GPF.<sup>4</sup> Management of the GPF should benefit from taking account of the notorious volatility of commodity prices. It has been argued that the effect of not holding oil and gas sector assets in the GPF would be negligible (Ministry of Finance, 2008), but the effect of leveraging up the weights of risky assets in the portfolio and hedging based on the correlation of oil was not considered. The issue is not mentioned at all in the current management mandate (NBIM, 2012). Instead, the GPF is focused on diversifying financial assets above the ground. Our arguments suggest that this leaves the government and people of Norway highly exposed to the volatility of oil prices and reserves below the ground.

The paper is laid out as follows. Section 2 revisits standard portfolio theory without any below-ground oil. Section 3 extends this by introducing a predetermined path for oil production, focusing on the cases where the oil price is completely spanned by the market (sections 3.1 and 3.2) and the case where it is not (section 3.3). Section 4 derives the optimal time path for oil extraction and shows that it is slowed down if oil prices are positively correlated with financial markets. Finally, section 5 concludes.

## 2. Portfolio allocation without oil

Following Merton (1990), suppose the policy maker chooses consumption  $C$  and assets weights  $w_i$ ,  $i = 1, \dots, n$ , to maximize the expected present value of utility with discount rate  $\rho > 0$ ,

$$(1) \quad J(F, t) = \max_{C, w_i} E_t \left[ \int_t^{\infty} U(C(s)) e^{-\rho(s-t)} ds \right],$$

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<sup>3</sup> At US\$700 billion the GPF is the largest single fund in existence. Evaluating governance, accountability and transparency, structure and behavior, GPF ranked first on the basis of the first two criteria and second overall, behind Alaska’s US\$45 billion permanent fund (Truman, 2008). It also receives the highest rating on the Linaburg-Maduell Transparency Index (SWF Institute, 2013). It has often been referred to as a “model” for managing sovereign wealth fund assets (Chambers, et al., 2012; Larsen, 2005).

<sup>4</sup> Norway has over 5 billion barrels of proven oil and 73 trillion feet of proven natural gas reserves (EIA, 2012). At 2012 prices these are worth US\$ 590 billion and US\$ 190 billion. Together they are similar in size to the GPF.

subject to the budget constraint

$$(2) \quad dF = \sum_{i=1}^m w_i (\alpha_i - r) F dt + (rF - C) dt + \sum_{i=1}^m w_i F \sigma_i dZ_i,$$

where the value function  $J(F, t)$  depends on the size of the sovereign wealth fund  $F$  and time  $t$ . The fund consists of  $m$  risky assets,  $i = 1, \dots, m$ , with drift  $\alpha_i$  and volatility  $\sigma_i$  and one safe asset,  $i = m+1$ , with return  $r$  and volatility  $\sigma_{m+1} = 0$ .<sup>5</sup> We define the total number of risky and safe assets as  $n \equiv m + 1$ . The fund is a portfolio consisting of  $N_i$  shares of assets,  $i = 1, \dots, n$ , each with price  $P_i$ , so that  $F = \sum_{i=1}^n P_i N_i$ . The share of each asset in the portfolio is defined as  $w_i \equiv P_i N_i / F$ , so that  $F = \sum_{i=1}^n w_i F$ . The price of each risky asset follows a Geometric Brownian Motion with constant coefficients,

$$(3) \quad dP_i = \alpha_i P_i dt + \sigma_i P_i dZ_i, \quad i = 1, \dots, m,$$

where  $dZ_i$  is a Wiener process with  $dZ_i dZ_j = \rho_{ij} dt$  and  $\rho_{ii} = 1$  for  $i = 1, \dots, m$ . The prices of the risky assets thus have covariance matrix  $\Sigma = [\sigma_{ij}]$  with  $\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j$ . We abstract from mean reversion in asset prices.

With complete markets the weight of the safe asset in the fund,  $w_n = 1 - \sum_{i=1}^m w_i$ , can be positive or negative corresponding to whether the weight of the risky portfolio in the fund is smaller or greater than one. This is known as taking a long position ( $w_n > 0$ ) or short position ( $w_n < 0$ ) in the safe asset. We call total holdings of risky assets the “portfolio”,  $(1 - w_n)F = \sum_{i=1}^m w_i F$ , and denote the share by  $w \equiv 1 - w_n$ .

For simplicity, we suppose that preferences exhibit constant relative risk aversion,  $U(C) = C^{1-\theta} / (1-\theta)$ , where  $\theta$  denotes the coefficient of relative risk aversion,  $1 + \theta$  the coefficient of relative prudence (Kimball, 1990) and  $1/\theta$  the coefficient of intertemporal substitution.<sup>6</sup> These preferences are part of the class of hyperbolic absolute risk aversion preferences, which permit explicit analytical solutions to the asset allocation problem (Merton, 1971). Solving the Hamilton-Jacobi-Bellman equation we obtain the optimal size of the “portfolio” and the allocation of risky assets within this “portfolio” (Merton, 1990).

**Proposition 1:** *In the absence of oil, the share of each risky asset in the “portfolio” is*

<sup>5</sup> Our analysis is partial equilibrium so the fund is treated as total wealth. This implicitly supposes that the government runs a non-fund balanced budget, so that can be abstracted from taxes and non-fund income.

<sup>6</sup> One can use Epstein-Zin preferences to separate out risk aversion and intertemporal substitution (Epstein and Zin, 1989) as has been done in similar continuous-time problems before (Attanasio and Weber, 1989; Wang et al., 2013).

$$(4) \quad w_i = \delta_i w, \quad \delta_i \equiv \frac{1}{v} \sum_{j=1}^m v_{ij} (\alpha_j - r).$$

and the share of the sum of all risky assets in the total portfolio is

$$(5) \quad w = \frac{v}{\theta}, \quad v \equiv \sum_{i=1}^m \sum_{j=1}^m v_{ij} (\alpha_j - r),$$

where  $v_{ij} \equiv [\Sigma^{-1}]_{ij}$ . The share of safe assets in the total portfolio equals  $1 - w$ .

The price per share of the optimal “portfolio”, denoted by  $P$ , follows the Geometric Brownian Motion

$$(6) \quad dP = \alpha P dt + \sigma P dZ,$$

where  $\alpha \equiv \sum_{i=1}^m \delta_i \alpha_i$ ,  $\sigma \equiv \sqrt{\sum_{i=1}^m \sum_{j=1}^m \delta_i \delta_j \sigma_{ij}}$  and  $dZ \equiv \frac{1}{\sigma} \sum_{i=1}^m \delta_i \sigma_i dZ_i$ .

**Proof:** See Merton (1990, chapter 5).

This highlights the Tobin-Markowitz separation theorem, which states that the multi-asset problem can be converted into a two-asset problem with one risky asset (the “portfolio”) and one safe asset.

First, the size of the risky “portfolio” is determined by (4) and is proportional to the overall risk-adjusted return of the portfolio  $v$  and the willingness to take risk (i.e., the inverse of the coefficient of relative risk aversion  $\theta$ ). It is independent of total wealth. If there is only one risky asset, (4) reduces to the Sharpe ratio,  $w = (\alpha_1 - r) / \theta \sigma_1^2$ , so that the “portfolio” is proportional to the excess return of the risky asset over the safe asset and inversely proportional to the coefficient of relative aversion and the variance of the return on the risky asset. With various risky assets the overall risk-adjusted return is lower if the risky assets are positively correlated with each other, so that there is less scope for fluctuations to offset each other and to hedge positions.

Second, the allocation of risky assets in the “portfolio” is independent of the degree of risk aversion and depends on the excess returns and the covariance structure of asset returns in the usual way. The “portfolio” will be diversified, so that a particular asset will have a higher weight if it is less correlated with other assets (i.e., lower  $v_{ij}$ ) as indicated by (5).

Since (6) indicates that the “portfolio” follows a Geometric Brownian Motion, we can show residual volatility is managed through precautionary saving and consumption is a constant proportion of the fund.

**Proposition 2:** Without oil the rate of change of consumption is



$$(7) \quad \frac{\frac{1}{dt}E_t[dC]}{C} = \frac{r - \rho}{\theta} + \frac{1}{2}(1 + \theta)\sigma^2 w^2,$$

the rate of consumption is

$$(8) \quad C = r^* F, \quad r^* \equiv \frac{\rho}{\theta} + \left(1 - \frac{1}{\theta}\right) \left[ r + \frac{1}{2\theta} \left( \frac{\alpha - r}{\sigma} \right)^2 \right],$$

and the fund follows the Geometric Brownian Motion

$$(9) \quad dF = \alpha^* F dt + \sigma w F dZ, \quad \alpha^* \equiv (\alpha - r)w + r - r^*,$$

which implies that

$$(10) \quad F(t) = F(0) \exp \left[ \left( \alpha^* - \frac{1}{2} \sigma^2 w^2 \right) t + \sigma w Z(t) \right], \quad t \geq 0.$$

**Proof:** Based on Merton (1990, chapter 5).

Aggregate risk is managed by precautionary saving (cf., Leland, 1968; Sandmo, 1970; Zeldes, 1986; Carroll and Kimball, 2008). This can be seen from the extra term on the right-hand side of the stochastic Euler equation (7), which indicates that the expected time path of consumption is tilted upwards. The degree of tilt increases with the coefficient of relative prudence  $1 + \theta$ , the riskiness of the “portfolio”  $\sigma^2$ , and the size of the risky portfolio,  $w$ . Precautionary saving will thus build up a buffer stock of assets by depressing consumption today. The buffer stock is not used to temporarily support consumption when asset prices are low, as asset price shocks are random walks and so are completely persistent. Instead, its sole function is to compensate future periods for bearing additional risk.

Equation (8) shows that consumption is a constant proportion of fund wealth. The marginal propensity to consume is affected by a higher return on the safe asset in two ways: an intertemporal substitution effect (negative as future consumption has become cheaper) and an income effect (positive as lifetime wealth has gone up). The intertemporal substitution effect dominates the income effect if the elasticity of intertemporal substitution,  $1/\theta$ , exceeds one. It can be seen from (8) that the marginal propensity to consume,  $r^*$ , then decreases with the return on the safe asset,  $r$ , and the average excess return on risky assets,  $\alpha - r$ ; and increases with the coefficient of relative risk aversion,  $\theta$ , and the volatility of the fund,  $\sigma$ .

The degree of precautionary saving depends on aggregate risk, as individual risky assets will hedge one another in the “portfolio”. If there is perfect positive or negative correlation between the risky assets, the

“portfolio” can be constructed so that all shocks offset each other and  $dZ = 0$  and there is no need for precautionary saving. However, this would not be optimal as the government is willing to accept some risk for a higher return. The sovereign wealth fund consists of the safe assets and the risky portfolio, where the risk of the driven by the aggregate risk of the “portfolio”,  $dZ$  in (9). Consumption is a constant fraction of the fund so that, as the fund grows through capital gains, consumption grows in a way that is consistent with precautionary saving. This results from absolute risk aversion,  $\theta/C$ , falling as consumption rises.

### *Norway's sovereign wealth fund in practice*

In practice the management of Norway's Government Pension Fund Global (GPF) is well-described by the three key elements of propositions 1 and 2 (see [www.nbim.no](http://www.nbim.no)), despite abstracting completely from oil. First, the Norwegian Ministry of Finance dictates the mix between equity assets and bonds. At present the mix is about 60% equity and 40% bonds, but prior to 2009 it was about 40% equity and 60% bonds. When the mandate was changed the Ministry of Finance did not alter the benchmark within the equity portfolio. Interpreting equity as "risky" and bonds as "safe" assets, this mandate is consistent with equation (4). Second, the optimal "portfolio" is designed by combining all assets in the market based on their risk, hedging potential and return properties. If all investors in the market have the same information about future asset prices, this can be interpreted as the "market portfolio" (Sharpe, 1964). This is consistent with Norway's use of the FTSE All Cap Index as the benchmark for their equity investments, which is in accordance with equation (5). Third, under Norway's budgetary rule or *handlingsregelen* the GPF releases 4% of the accumulated assets into the general budget each year. This is consistent with equation (8).

## **3. Portfolio allocation for a given path of oil extraction**

### **3.1. Complete markets: oil can be replicated by a bundle of one risky asset and the safe asset**

For an oil-rich economy total wealth consists of oil wealth and non-oil wealth. The stock of oil wealth cannot easily be traded, but it can be treated as tradable if its return can be replicated by a synthetic portfolio of traded assets. In that case, there are complete markets and one can offset oil or hedge oil in the fund. Let there be one single asset whose return is perfectly correlated with the oil price increment, so all hedging can be done with this single asset. Section 3.2 generalizes this to when oil price increments are perfectly spanned by all the other assets. Due to geological or engineering constraints oil production  $O$  is predetermined and constant for time  $t \in [0, T)$  and zero thereafter. Let  $P_O$  be the price of oil and suppose that the country is a small oil exporter that cannot influence the oil price. The policy maker then solves:

$$(1') \quad J(F, P_O, t) = \max_{C, w_i} E_t \left[ \int_t^\infty U(C(s)) e^{-\rho(s-t)} ds \right]$$

subject to the budget constraint

$$(2') \quad dF = \sum_{i=1}^m w_i (\alpha_i - r) F dt + (rF + P_O O - C) dt + \sum_{i=1}^m w_i F \sigma_i dZ_i$$

and the Geometric Brownian Process for asset prices (3) and the oil price

$$(11) \quad dP_o = \alpha_o P_o dt + \sigma_o P_o dZ_o,$$

where the drift in the oil price is not too large,  $\alpha_o < r$ .<sup>7</sup> Complete markets means that there exists a traded security, say asset  $k \in [1, m]$ , whose instantaneous return is perfectly correlated with oil income, so that  $dZ_o = dZ_k$ . The following proposition shows how oil can be replicated by a bundle of two assets.

**Proposition 3:** *If there is one financial asset  $k \in [1, m]$  whose return is perfectly correlated with oil returns, oil revenue can be exactly replicated by a bundle of this asset and the safe asset  $n$ . The value of this bundle is then the capitalized value of oil revenue:*

$$(12) \quad V(P_o, t) = P_o(t) O \left\{ 1 - \exp[-\psi_k(T-t)] \right\} \frac{1}{\psi_k}, \quad \psi_k \equiv r - \alpha_o + \beta_k(\alpha_k - r), \quad \beta_k \equiv \frac{\sigma_o}{\sigma_k}.$$

Total wealth consists of fund assets and subsoil oil assets,  $W = F + V$ , and behaves according to

$$(13) \quad dW = \sum_{i=1}^m \bar{w}_i W (\alpha_i - r) dt + (rW - C) dt + \sum_{i=1}^m \sigma_i \bar{w}_i W dZ_i,$$

where  $\bar{w}_i \equiv \frac{w_i F}{F + V}$  for  $i \neq k$  and  $\bar{w}_k \equiv \frac{w_k F + \beta_k V}{F + V}$ .

**Proof:** See appendix A.1.

This mimicking result only requires oil and asset  $k$  to be perfectly correlated, not to have the same variance, so that  $dZ_o = dZ_k$  but  $\sigma_o \neq \sigma_k$ . The bundle is constructed by buying a number of shares in the risky and safe asset and the price of the bundle is the value of oil wealth. The number of risky shares  $k$  is chosen so that the bundle has the same variance as oil and the number of safe shares is chosen so that the bundle has the same drift as the price of oil. Since oil wealth and the bundle behave identically and markets are complete, the price of the replicating bundle is the value of oil wealth (12). Oil wealth corresponds to the present value of oil revenues using the discount rate  $\psi_k$ , which has been adjusted for the market price of risk as risk-averse investors require compensation for bearing risk.<sup>8</sup>

Although oil production is predetermined, oil revenues can be treated as tradable because of the replicating bundle stated in proposition 3. We suppose that claims to oil cannot be packaged and sold off, because of political or practical constraints. If the claims to oil can be sold off, the proceeds could simply

<sup>7</sup> This is consistent with empirical estimates (e.g., van den Bremer and van der Ploeg, 2013).

<sup>8</sup> The value of an uncertain stream of income can also be found by discounting at the risk-free rate if the probability space is adjusted to a risk-neutral measure with the aid of a theorem due to Cameron, Martin and Girsanov (1960).

be invested in a diversified portfolio as described in section 2.<sup>9</sup> However, this is not necessary if oil can be replicated. Any net exposure to oil price risk,  $dZ_o$ , can be artificially constructed by combining oil revenue with an amount of the replicating bundle. The problem can thus be simplified into choosing the net weight of each risky asset,  $\bar{w}_i$  for  $i = 1, \dots, m$ , in total wealth,  $W = F + V$ . This is analytically identical to the problem of section 2 and gives rise to the next proposition which shows that the net exposure to each asset has to be the same constant share of total wealth as in proposition 1.

**Proposition 4:** *If there is one financial asset  $k \in [1, m]$  whose return is perfectly correlated with oil returns, the weight of each risky asset in total wealth will be constant:*

$$(14) \quad \bar{w}_i = \delta_i w, \quad i = 1, \dots, m, \quad \bar{w} \equiv \sum_{i=1}^m \bar{w}_i = w,$$

where  $w$  and  $\delta_i$  are defined in proposition 1. The weight of each risky asset in the fund is given by

$$(15) \quad w_i = \bar{w}_i \frac{F+V}{F}, \quad i \neq k, i = 1, \dots, m, \quad \text{and} \quad w_k = \bar{w}_k \frac{F+V}{F} - \beta_k \frac{V}{F}.$$

**Proof:** By direct analogy to proposition 1 and from the results of proposition 3.

Equation (15) shows that oil wealth creates, in addition to the demands for risky assets in the absence of oil ( $\bar{w}_i, i = 1, \dots, m$ ), both a leverage demand for each risky asset ( $(V/F)\bar{w}_i, i = 1, \dots, m$ ) and a negative hedging demand ( $-(V/F)\beta_k$ ) for the perfectly positively correlated (or replicating) risky asset. Note that these additional demands are proportional to the ratio of oil wealth to fund wealth. This means that the weight of every risky asset in the fund is leveraged by the ratio of total wealth to fund wealth, allowing for the additional oil wealth outside the fund. Hedging demand is proportional to the beta relevant for oil and the replicating asset,  $\beta_k = \sigma_o / \sigma_k > 0$ , and the leverage factor,  $V/F$ . Since asset  $k$  perfectly replicates oil, hedging demand is negative and oil risk can be fully offset by going short in this asset.

Over time oil wealth's effect on the fund will change as reserves are depleted. Initially there is a lot of exposure to oil price risk as a lot of oil remains in the crust of the earth. The weight of the safe asset,

$$w_n = 1 - \bar{w} + (\beta_k - \bar{w}) \frac{V}{F},$$

will therefore be relatively higher if negative hedging demand exceeds total demand for risky assets in the absence of oil,  $\beta_k > \bar{w}$ . As oil is depleted ( $V \rightarrow 0$ ) the leverage factor falls, the leverage demand for each risky asset falls back to their reference weight,  $w_i \rightarrow \bar{w}_i, i \neq k$  and

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<sup>9</sup> There is little evidence of countries selling all oil rights upfront. An initial payment for a well is often made by extracting firms as part of an auction process. Since these firms are risk averse, they will not be willing to take on all price and production risk.

$w_k \rightarrow \bar{w}_k$ , and the negative hedging demand for asset  $k$  tapers to zero. The weights of the fund should thus be continuously adjusted until oil has run out. In particular, as oil is extracted and the fund matures, assets should be reallocated from risky towards safe assets if  $\beta_k > \bar{w}$ .<sup>10</sup>

**Figure 1: Portfolio weights with different correlations structures of risky assets with oil**

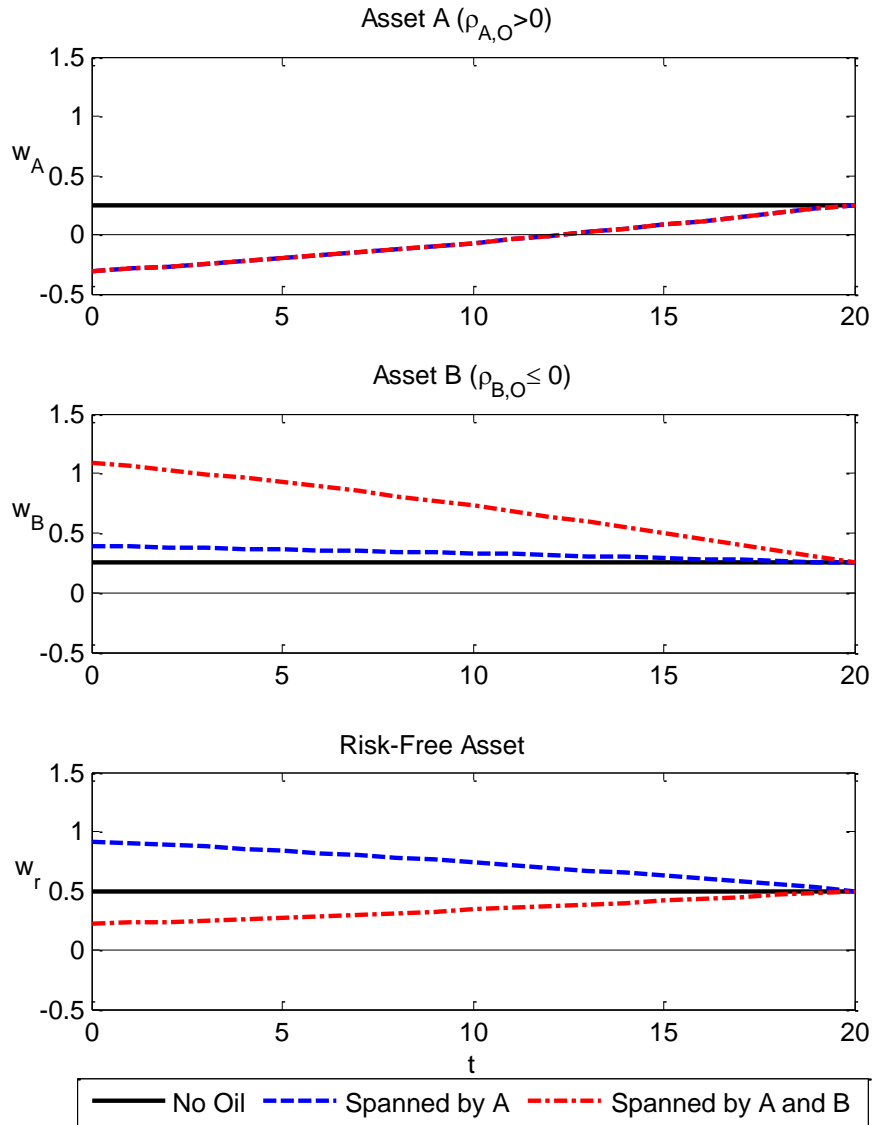


Figure 1 shows what happens with complete markets and a fund consisting of two risky assets  $A$  and  $B$  and a safe asset.<sup>11</sup> In the “Spanned by  $A$ ” case, the return on asset  $A$  is perfectly correlated with oil and

<sup>10</sup> This assumes that withdrawals from the fund are not so rapacious (i.e.,  $\rho$  is not too high, cf. (8)) that fund assets fall quicker than oil is extracted and  $V/F$  rises over time.

that on asset  $B$  is independent of both asset  $A$  and oil. Initially, Asset  $B$  is leveraged and has a higher weight than without oil. This is also the case for asset  $A$ , but this extra demand is more than offset by negative hedging demand so that oil is offset by a short position in asset  $A$ . Over time oil wealth declines and leveraging fades out, so that the weight of asset  $B$  drops to its non-oil weight. That of asset  $A$  increases to its non-oil weight as both the negative hedging demand and the leverage demand taper off. The share held in the safe asset is initially higher to allow for the short position in asset  $A$ , but tapers off to the non-oil share as oil reserves are depleted.

Norway is approaching the end of its oil reserves with a reserves-to-production ratio of 9.2 (BP, 2012). Our results suggest that Norway should have held a large offsetting position in oil securities (e.g., futures and forward contracts) and invested in other risky assets more aggressively. As oil is further depleted during the decades ahead, the portfolio shares should be adjusted towards a more even allocation. In contrast, Kuwait with a reserves-to-production ratio of nearly 100 can justify a much bigger proportion of risky assets in their fund.

The following proposition gives the effects of oil wealth on consumption.

**Proposition 5:** *If there is one financial asset whose return is perfectly correlated with oil returns, the rate of change of consumption is given by*

$$(16) \quad \frac{\frac{1}{dt} E_t [dC]}{C} = \frac{r - \rho}{\theta} + \frac{1}{2} (1 + \theta) \sigma_w^2 \bar{w}^2,$$

the rate of consumption is

$$(17) \quad C = r_w^* W, \quad r_w^* \equiv \left(1 - \frac{1}{\theta}\right) r + \frac{\rho}{\theta} + \frac{1}{2\theta} \left(1 - \frac{1}{\theta}\right) \left(\frac{\alpha_w - r}{\sigma_w}\right)^2.$$

where the drift and the volatility of total wealth,  $\alpha_w$  and  $\sigma_w$  correspond to  $\alpha$  and  $\sigma$  in proposition 1.

Total wealth follows the Geometric Brownian Motion

$$(18) \quad dW = \alpha_w^* W dt + \sigma_w \bar{w} W dZ_w, \quad \alpha_w^* \equiv (\alpha_w - r) \bar{w} + r - r_w^*,$$

where aggregate volatility is  $dZ_w = \sum_{i=1}^m \delta_i \frac{\sigma_i}{\sigma_w} dZ_i$ . This implies that

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<sup>11</sup> The following illustrative figures assume that  $F(0)=100$ ;  $r=\rho=0.05$ ;  $\theta = 0.5$ ;  $P_i(0) = 1$ ;  $\alpha_i = 0.07$ ;  $\sigma_i = 0.02$ ;  $\rho_{ij} = 0$ ;  $O = 5$ ;  $\alpha_o = 0.01$  and  $\sigma_o = 0.25$ .

$$(19) \quad W(t) = W(0) \exp \left[ \left( \alpha_w^* - \frac{1}{2} \sigma_w^2 \bar{w}^2 \right) t + \sigma_w \bar{w} Z_w(t) \right], \quad t \geq 0.$$

**Proof:** By direct analogy to proposition 2.

Consumption is thus postponed to manage the residual volatility of all assets,  $\sigma_w^2 \bar{w}^2$ . Oil price shocks are fully hedged within the fund portfolio, so the government is only exposed to aggregate volatility,  $dZ_w$ . Consumption will be a constant proportion of total wealth. With no uncertainty (17) states that consumption is a constant proportion of total fund wealth and oil wealth. This echoes the Hartwick rule which states that any running down of below-ground oil wealth must be exactly compensated for by an equal buildup of above-ground financial assets so that total wealth is unaffected and consumption is fully smoothed (Hartwick, 1977). With uncertainty in (17) this hypothesis is adjusted for the risk and return on total wealth. With a proper construction of the fund consumption should not be directly affected by oil price shocks, only indirectly through their effect on total wealth.

**Figure 2: Optimal time paths of oil, assets and consumption**

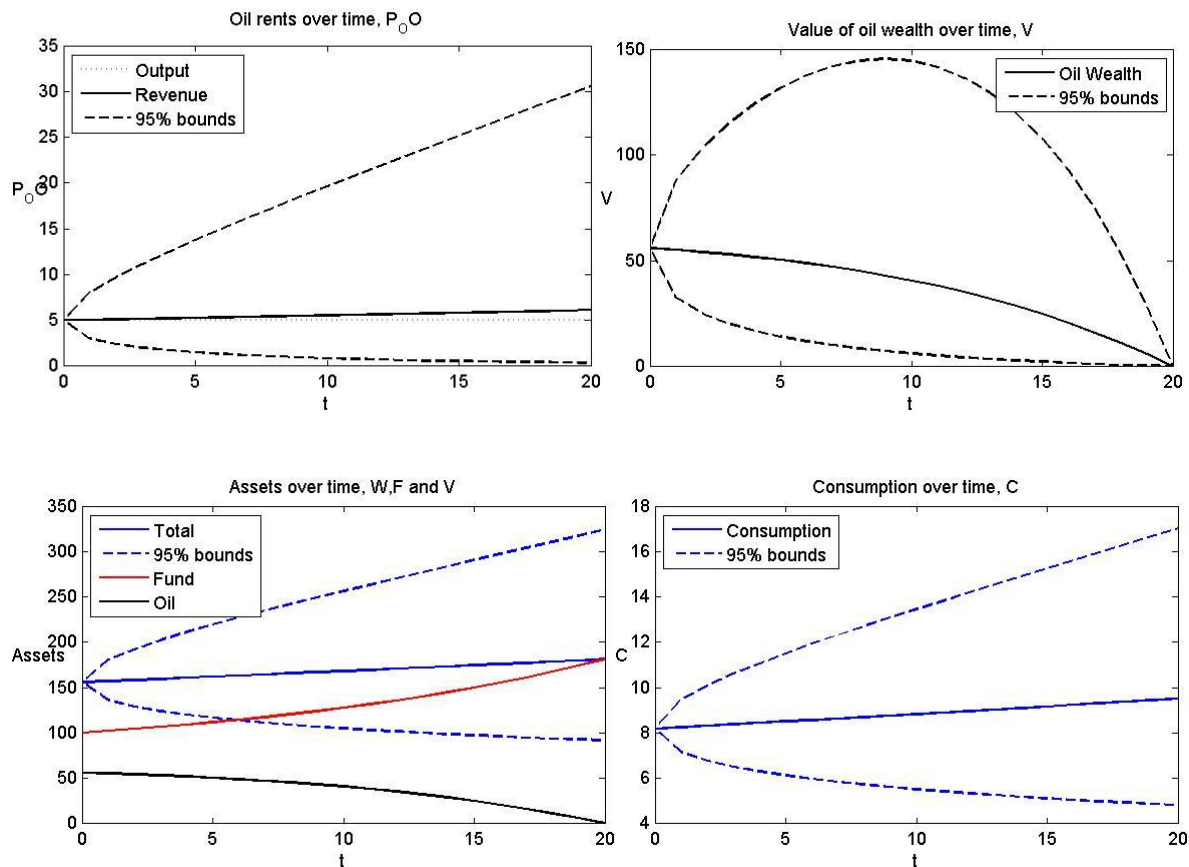




Figure 2 illustrates the paths of oil, assets and consumption. The drift in oil revenue is not too big,  $\alpha_o < r$ , and oil revenue is more uncertain in the future (see panel (a)). The value of oil decreases with time as more reserves have been extracted (see panel (b)). The volatility of remaining oil wealth also declines after a point, because less remains exposed to volatile oil prices. Since oil price uncertainty is hedged within the portfolio, the policy maker is only exposed to aggregate volatility of total wealth. Consuming a constant proportion of total wealth and precautionary saving imply that total wealth and consumption steadily rise as oil wealth is extracted and converted into financial wealth (see panels (c) and (d)).

### 3.2. Complete markets: oil can be replicated by a combination of all assets

There may not be a single asset that is perfectly correlated with the oil price. However, complete markets still prevail if oil price increments can be perfectly spanned by all other financial assets. A way to capture this is to suppose that all risky assets are driven by a common set of underlying shocks (e.g., to demand, supply, technology or the weather),  $du \sim \text{i.i.d. } N(0, dt)$ . Each asset may be affected by these underlying shocks differently, which gives rise to the correlations  $dZ = \Lambda du$ , where  $\Lambda = [\lambda_{ij}]$  is an invertible matrix of  $m \times m$  constants and  $dZ = [dZ_1, \dots, dZ_m]'$  is the vector of (possibly correlated) Wiener processes driving the returns on the risky financial assets. Complete markets implies that oil can be spanned by all financial assets, so that the Wiener process driving oil prices can be written as:

$$(20) \quad dZ_o = \Lambda_o du = \Lambda_o \Lambda^{-1} dZ,$$

where  $\Lambda_o = [\lambda_{o1}, \dots, \lambda_{om}]$  is a  $1 \times m$  vector of constants determining how the oil price responds to the vector of underlying shocks,  $du$ .

**Proposition 6:** *If oil returns are perfectly spanned by all other financial assets, oil revenue can be replicated by a particular bundle of all risky financial assets in the market,  $i = 1, \dots, m$ , and the safe asset,  $n$ . The value of this bundle is the capitalized value of oil revenues:*

$$(12') \quad V(P_o, O, t) = P_o(t) O \left\{ 1 - \exp[-\psi(T-t)] \right\} \frac{1}{\psi}, \quad \psi \equiv r - \alpha_o + \sum_{i=1}^m \beta_i (\alpha_i - r), \quad \beta_i \equiv \frac{M_i \sigma_o}{\sigma_i},$$

where  $M \equiv [\Lambda_o \Lambda^{-1}]_i$ . Total wealth consists of fund assets and subsoil oil assets,  $W = F + V$ , and evolves according to (13) with  $\bar{w}_i \equiv \frac{w_i F + \beta_i V}{F + V}$ ,  $i = 1, \dots, m$ .

**Proof:** See appendix A.1.

Full hedging of oil wealth now requires a bundle of all assets. This is possible as the underlying shocks affecting the oil price will also affect all the risky financial assets in the market.<sup>12</sup> The oil price can be replicated by linearly combining small exposures to many financial assets, as they are all driven by correlated and normally distributed processes. These exposures depend on the similarity of each financial asset to the oil price and its uniqueness,  $\beta_i$ . We call this the “replicating bundle”. To ensure that the net exposure to each financial asset is a constant share of total wealth as shown in (14), proposition 6 implies that their weight in the fund consists of an additional leverage and a hedging component:<sup>13</sup>

$$(15') \quad w_i = \underbrace{\bar{w}_i + \bar{w}_i \frac{V}{F}}_{\text{leveraged demand}} + \underbrace{\left( -\beta_i \frac{V}{F} \right)}_{\text{hedging demand}}, \quad \beta_i = \frac{M_i \sigma_o}{\sigma_i}, \quad i=1, \dots, m.$$

Both the leverage and the hedging demands increase with the ratio of oil wealth to fund wealth. It is thus higher when there is still a lot of oil in the ground. As oil is depleted with the passage of time, the leverage factor falls and thus the leveraged demand falls to its non-oil level and hedging demand vanishes. With one risky financial asset,  $m = 1$ , the total leveraged demand for risky assets can be written

as  $\left( \frac{\alpha_1 - r}{\theta \sigma_1^2} \right) \left( \frac{F + V}{F} \right)$  and hedging demand as  $-\frac{\rho_{o1} \sigma_o}{\sigma_1} \frac{V}{F}$ , where  $\rho_{o1}$  is the correlation coefficient

between oil and risky asset returns.<sup>14</sup> We thus see that the total leveraged demand for the risky asset increases with the excess return and the oil leverage, but decreases with the riskiness of the risky asset and relative risk aversion. This leveraged demand is thus a leveraged version of the Sharpe ratio.

The hedging demand for each asset also increases with the ratio of oil reserves to fund wealth. With one risky asset hedging demand is thus the product of the leverage and the oil-asset beta, where the latter corresponds to the slope coefficient of a regression of demeaned asset returns against demeaned oil returns. If oil price risk is purely idiosyncratic ( $\rho_{o1} = 0$ ), hedging demand is zero. Positive correlation of the financial asset and oil ( $\rho_{o1} > 0$ ) increases the volatility of total wealth, so that going short in the risky asset (i.e., negative hedging demand) helps to curb volatility of total wealth. Negative correlation of the asset ( $\rho_{o1} < 0$ ) and oil requires, in contrast, an even bigger allocation to the risky financial asset to hedge oil price risk (positive hedging demand). Equation (15') generalizes this analysis to multiple risky financial assets.

<sup>12</sup> One could also use a general equilibrium model to show how oil price fluctuations are driven by more fundamental shocks (e.g., Bodenstein et al., 2012).

<sup>13</sup> Merton (1990, chapter 21) refers to these components as a “wealth” effect and a “substitution” effect.

<sup>14</sup> A similar expression can be obtained by mean-variance analysis (Gintschel and Scherer, 2008; Scherer, 2009).

If all financial asset returns are independent of each other (i.e., the matrix  $\Lambda$  is diagonal), one should hedge oil by investing more in assets that are negatively correlated with oil (e.g., assets that use oil as an input such as manufacturing and consumer goods industries) and less in assets that are substitutes for oil (e.g., renewable energy), especially when there is still a lot of oil in the ground and exposure to oil price risk is high. In that case, one should also leverage up all the demand for risky assets that would prevail in the absence of oil.

Figure 1 also offers a simple illustration for the case of two risky assets imperfectly correlated with oil: asset  $A$  positively ( $\rho_{OA} = 0.5$ ), asset  $B$  negatively ( $\rho_{OB} = -0.5$ ), and returns on the two assets independent. We observe from comparing the two cases in figure 1 that oil wealth is now hedged by investing less in the asset that is positively correlated with oil,  $A$ , and more in the asset that is negatively correlated with oil,  $B$ . To make this possible, less of the safe asset must be held. The leverage effect increases the holdings of  $A$  and  $B$  relative to the case without oil. Over time the fund becomes a larger share of total wealth and the asset shares converge to what they would be without oil.

More generally, hedging demand depends on the covariance of each financial asset with oil and with each other. For example, consider an underlying shock  $du_G$  which affects oil and asset  $A$  but no others, i.e.,  $\lambda_{OG}, \lambda_{AG} > 0$  and  $\lambda_{iG} = 0$ , for all  $i = 1, \dots, m$ . All other underlying shocks  $du_j$  affect oil and asset  $A$  in opposite ways, so  $\lambda_{Oj} > 0$  and  $\lambda_{Aj} < 0$ , for all  $j = 1, \dots, m$ . In this case, it is possible that oil and asset  $A$  are negatively correlated,  $\sum_{j=1}^m \lambda_{Oj} \lambda_{Aj} < 0$ , but the fund should nevertheless invest *less* in asset  $A$  to offset the exposure to shock  $G$ . The allocation of all other assets will have to adjust to hedge the effects of the remaining shocks,  $du_j$  for  $j \neq g$ .

### 3.3. Incomplete markets

To capture that there may be a part of oil returns that is not correlated in any way with the market we generalize the Wiener process driving oil returns to:

$$(20') \quad dZ_O = \Lambda_O \Lambda^{-1} dZ + \lambda_{O0} du_0, \quad E[du_0 dZ_i] = 0, \quad i = 1, \dots, m,$$

where  $\lambda_{O0}$  is a scalar constant and  $du_0$  is an independent Wiener process which is uncorrelated with the Wiener processes driving the returns on risky financial assets. The parameters  $\Lambda_O$ ,  $\Lambda$  and  $\lambda_{O,0}$  can be estimated by applying principal components analysis to the covariance matrix of the financial assets with

oil. When a component of the oil price is unspanned by financial assets, there will be bigger need for precautionary saving to cope with residual volatility.<sup>15</sup>

**Proposition 7:** *With incomplete markets characterized by (20') the stochastic Euler equation can be approximated by*

$$(21) \quad \frac{\frac{1}{dt} E_t [dC]}{C} = \frac{r - \rho}{\theta} + \frac{1}{2}(1 + \theta) \left[ \sigma^2 w^2 + \lambda_{o,0}^2 \sigma_o^2 \left( \frac{V}{W} \right)^2 \right],$$

where  $w$  and  $\sigma$  are defined in proposition 1.

**Proof:** See appendix A.2.

The first term on the right-hand side of (21) is the usual slope of the optimal consumption profile in deterministic settings. The second term on the right-hand side of (21) captures precautionary saving and is therefore proportional to the coefficient of relative prudence,  $CRP = 1 + \theta$ . The term  $\sigma^2 w^2$  inside the square brackets arises from the precautionary saving that is needed under complete markets where all oil price volatility can be fully diversified. It is proportional to the variance of the portfolio of risky financial assets and the share of risky assets in the fund squared. The other term inside the square brackets is

$\lambda_{o,0}^2 \sigma_o^2 \left( \frac{V}{W} \right)^2$  and arises from the precautionary saving that is required because not all oil price volatility

can be fully hedged. Less spanning of the oil price corresponds to a higher  $\lambda_{o,0}$  and thus implies that more precautionary saving is required, especially if the share of oil wealth to total wealth ( $V/W$ ) and the volatility of the oil price are high. Hence, compared with the situation where oil prices and the fund are perfectly correlated and markets are complete, the consumption path becomes steeper and thus more precautionary saving is needed. This effect is larger the less the oil price is spanned by financial asset prices and the larger oil price volatility, but the effect evaporates as depletion leads to the share of oil wealth in total wealth fading away.

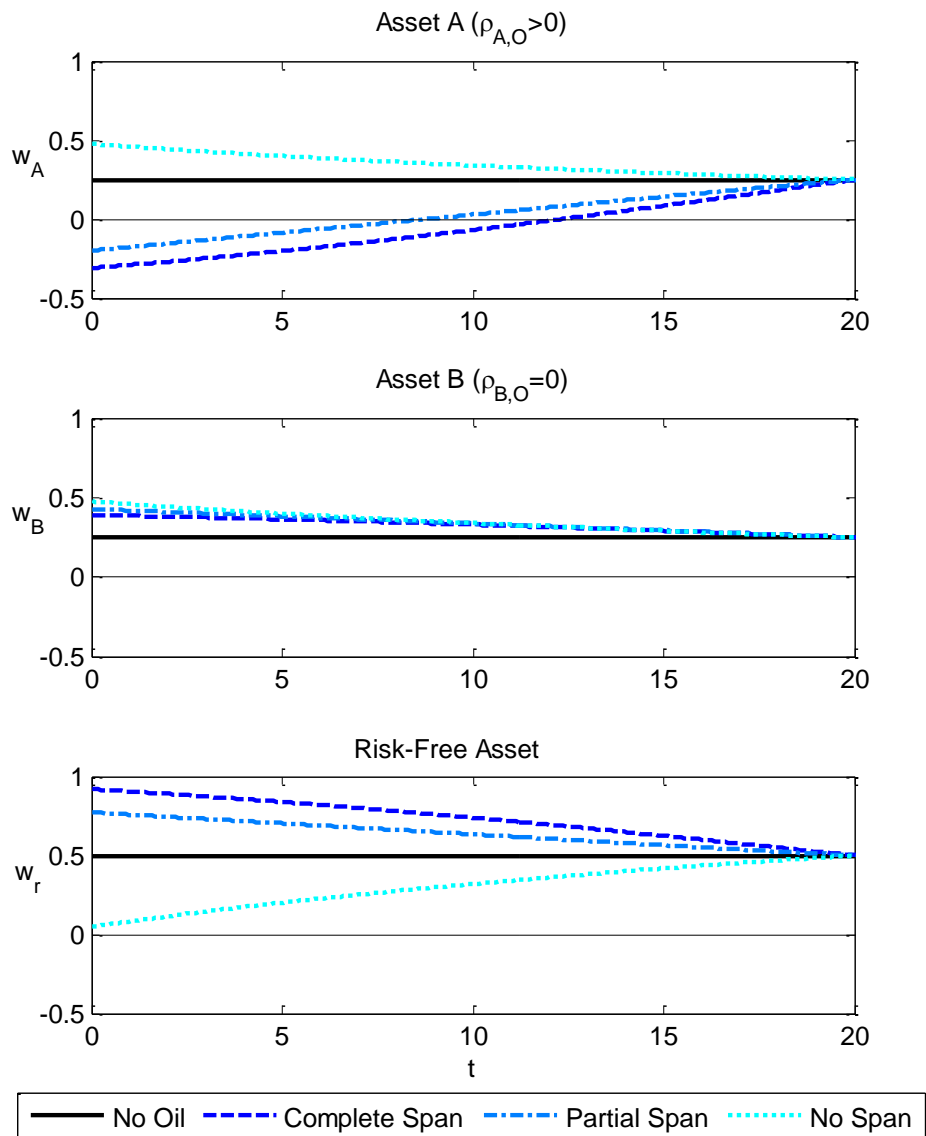
To illustrate this additional precautionary saving, we take the two-asset example of figure 1 with  $A$  perfectly correlated with the oil price and  $B$  not at all, and add non-diversifiable risk. The case of complete spanning (dark blue) corresponds to that in figure 1. Holdings of both financial assets are leveraged up, especially when oil wealth is highest. In addition, there is hedging demand for the perfectly

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<sup>15</sup> Earlier work abstracted from risky financial assets and focused at oil price volatility only (an extreme case of incomplete markets) to show that a fund is needed to smooth the benefits of oil extraction over different generations and to cushion against adverse oil price shocks (van den Bremer and van der Ploeg, 2013). Here we extend the analysis to allow for risky assets too, but still remaining within the realm of incomplete markets.

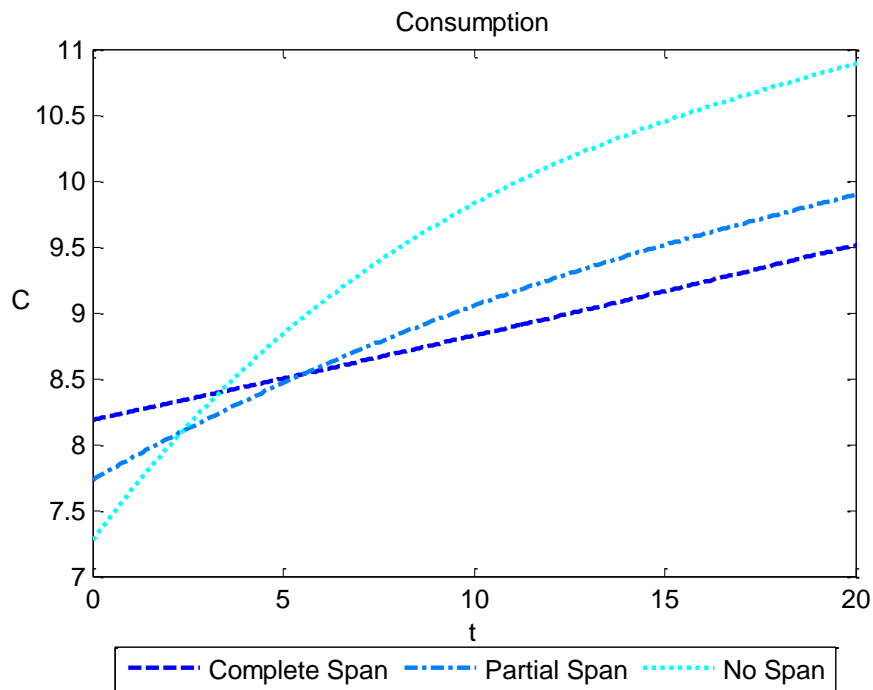
correlated asset  $A$  to offset oil price volatility. In the beginning the holdings of asset  $A$  are negative, implying a short position (borrowing) in the asset, to invest in the risk-free asset. Over time the holdings of both risky assets,  $A$  and  $B$ , are unwound into the shares that prevail in the absence of oil. Effectively, holdings of financial assets are deleveraged as oil reserves are depleted. Consumption rises over time as wealth rises. This is a consequence of using precautionary savings to manage the residual volatility of total wealth.

**Figure 3: The effect of non-diversifiable risk on portfolio shares**



The case of no spanning at all (the dotted lines) assumes that the oil price is uncorrelated with both risky assets. This results in a leveraged demand for both assets without any offsetting hedging demand. All oil price volatility will be unhedged, and so must be managed by a large degree of precautionary saving. The case of partial spanning is like the example given in figure 1 but with residual oil price volatility added. This combines the cases of complete and zero spanning, leading more precautionary saving and a steeper consumption path.

**Figure 4: The effect of non-diversifiable risk on consumption**



#### 4. Portfolio allocation with endogenous oil extraction

We now turn our attention to how financial assets should affect oil extraction. The workhorse of resource economics is the Hotelling rule, which states that the return on keeping oil in situ (the expected capital gains) must equal the return of extracting oil, selling it and getting a return on it (the return on the safe asset) (Hotelling, 1931). This rule then dictates the optimal speed of extracting oil from the earth. We now illustrate how this canonical rule has to be modified in the presence of both volatile oil and financial asset prices. To tackle this question we assume that markets are complete so that there exists a traded asset  $k$  which is perfectly correlated with oil,  $dZ_k = dZ_o$  (as in section 3.1). We will find that oil wealth has to be hedged by the fund, which involves dynamically allocating assets to ensure that the net exposure to oil

remains a constant share of total wealth. Consumption should not be affected by the path of oil extraction, only by the way the path of oil extraction changes the present value of oil wealth. If marginal oil rents are positively correlated with other assets, we find that the rate of extraction will be slower on average than predicted by the standard Hotelling rule.<sup>16</sup> This leaves a buffer of oil beneath the ground which can be extracted when oil prices and thus marginal oil rents are high, which can then be invested on the stock market to obtain on average higher returns.

#### 4.1. Optimal rates of oil extraction

The policy maker chooses consumption  $C$ , the rate of oil extraction  $O$ , and asset weights  $w_i$ ,  $i=1, \dots, m$  to maximize expected welfare

$$(1'') \quad J(F, P_O, S, t) = \max_{C, w_i, O} E_t \left[ \int_s^\infty U(C(s)) e^{-\rho(s-t)} ds \right]$$

subject to the budget constraint

$$(2'') \quad dF = \sum_{i=1}^m w_i (\alpha_i - r) F dt + [rF + \Omega(P_O, O) - C] dt + \sum_{i=1}^m w_i F \sigma_i dZ_i,$$

the Geometric Brownian Processes for asset prices (3) and the oil price (11) and the depletion equation

$$(22) \quad dS = -O dt,$$

where oil rents are revenues less extraction costs,  $\Omega(P_O, O) \equiv P_O O - G(O)$ , and total extraction costs are increasing in the extraction rate ( $G' > 0$ ) and convex ( $G'' < 0$ ) (cf., Pindyck, 1984). Cumulative oil extraction cannot exceed given initial oil reserves,  $\int_0^\infty O(t) dt \leq S_0$ . In practice, oil fields evolve stochastically as new fields are discovered and existing fields becomes more or less economical (e.g., Pindyck, 1978). They are also endogenous to exploration effort, but we abstract from these complications here.

**Proposition 8:** *The optimal path for the expected rate of oil extraction satisfies*

$$(23) \quad \frac{1}{dt} E[d\Omega_O] = r\Omega_O - \frac{-\frac{1}{dt} E[dJ_F d\Omega_O]}{J_F(F, P_O, S, t)}.$$

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<sup>16</sup> Following the literature we assume that oil extraction can be adjusted instantaneously. In practice oil extraction is less flexible than asset portfolios, though we abstract from this distinction.

Under the assumption of quadratic extraction costs,  $G(O) = \gamma O^2 / 2$ ,  $\gamma > 0$ , the path for the actual rate of oil extraction  $O = O(F, P_O, S, t)$  satisfies

$$(24) \quad d\Omega_O = r\Omega_O dt + \gamma O_F \sum_{i=1}^m [(\alpha_i - r)dt - \sigma_i dZ_i] w_i F - (1 - \gamma O_F) [(\alpha_k - r)dt - \sigma_k dZ_O] P_O \frac{\sigma_O}{\sigma_k}.$$

Using the partial  $O_F$  from the deterministic solution, the stochastic path for the rate of oil extraction and the expected rate of oil extraction are approximately given by:

$$(25) \quad dO \cong rO dt - \frac{1}{\gamma} P_O(t) \sigma_O \left[ \left( \frac{r - \alpha_O}{\sigma_O} - \frac{\alpha_k - r}{\sigma_k} e^{-(r - \alpha_O)(T-t)} \right) dt - \left( 1 - e^{-(r - \alpha_O)(T-t)} \right) dZ_O \right],$$

$$(26) \quad E[O(t)] = \frac{1}{\gamma} E[P_O(t)] \left[ 1 - \left\{ 1 + \left( \frac{\alpha_k - r}{\alpha_k} \right) \sigma_O \right\} e^{-(r - \alpha_O)(T-t)} \right].$$

**Proof:** See appendix A.3.

The first-term on the right-hand side of (23) corresponds to the standard Hotelling rule, which states that the expected rate of change of marginal oil rents must equal the return on safe assets. In the absence of stochastic shocks this is the rule that prevails. Provided  $\alpha_O < r$  the rate of oil extraction will decline over time. The second term on the right-hand side of (23) is negative if marginal oil rents are positively correlated with the value of the financial assets in the fund, because a higher marginal oil rent is then associated with a higher value of the fund  $F$  and thus with a lower marginal utility from an extra dollar in the fund (i.e.,  $\frac{1}{dt} E[dJ_F d\Omega_O] < 0$ ). Equation (23) implies that, if oil is pro-cyclical, the rate of oil extraction is slower than predicted by the deterministic Hotelling rule (as  $\frac{1}{dt} E[d\Omega_O] < r\Omega_O$ ).

Intuitively, the marginal benefit of extracting oil is the rent it generates multiplied by the utility from investing oil rents in the fund,  $\Omega_O J_F$ .<sup>17</sup> This marginal benefit must equal the marginal benefit from keeping the oil in the ground,  $J_S$ . Smoothing extraction costs of oil over time requires  $\frac{1}{dt} E[dJ_S] = 0$ . The marginal utility from investing in the fund falls at the rate equal to the safe rate of return,  $\frac{1}{dt} E[dJ_F] = -rJ_F$ . Hence, to keep everything in line the marginal oil rent must grow at the safe rate of return,  $J_S = \Omega_O J_F$ . However, if marginal oil rents and investments in the fund move together an

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<sup>17</sup> This also equals the marginal utility of consumption,  $J_F = e^{-\rho t} U'(C)$ .



adjustment has to be made to slow the rate of oil extraction. Else, the benefit from extracting oil will outpace the benefit from leaving it in the ground.

Equation (25) indicates that the rate of oil extraction is positively correlated with the oil price. A sudden jump in the oil price then also requires a jump in the rate of oil extraction to make the most of it. The reason is that increasing the rate of oil extraction increases marginal extraction cost (as  $G'' > 0$ ), which limits the jump in marginal rents ( $\Omega_o(P_o, O) = P_o - G'(O)$ ). Clearly, an oil price shock affects the rate of oil extraction most when reserves are highest, since then most is still exposed to oil price risk. As the date of exhaustion approaches, the path of the rate of oil extraction gets closer to what it would have been in the absence of volatile oil and asset prices.

Equation (26) indicates that the expected rate of oil extraction is proportional to the expected oil price and inversely proportional to the marginal cost of extraction parameter  $\gamma$ . Without oil price uncertainty (26)

shows that the rate of oil extraction declines over time, since  $E[O(t)] = \frac{1}{\gamma} P_o(t) [1 - e^{-(r-\alpha_o)(T-t)}]$ . If the

asset that replicates the oil price has a higher expected return than the return on the safe asset,  $\alpha_k > r$ , then (26) implies that oil price uncertainty depresses the rate of oil extraction.

To illustrate our core result of slowing down of oil extraction, consider the setting of two assets  $A$  and  $B$  with  $A$  being perfectly correlated with oil and  $B$  not all (cf. figure 1). Figure 5 then confirms that procyclical oil prices delays oil extraction on average.<sup>18</sup> The point is that this leaves somewhat more oil in the ground, to be extracted when the oil price is high. Extracting later has two benefits: oil is sold for a high price, and those revenues can be invested for a higher return. If oil and the stock market were independent, the extraction path would only depend on the expected path of oil prices. Only if oil and the market interact will there be an additional effect on the rate of oil extraction.

Our finding that stochastic oil price shocks slow down the rate of oil extraction should be contrasted with earlier studies which find the opposite, in a framework without investment in financial assets (Pindyck, 1981; van der Ploeg, 2010). The reason is “extractive prudence” which is driven by sufficiently convex marginal extraction costs,  $G''' > 0$ .<sup>19</sup> Hence, it is better to extract oil more quickly because once it is out of the ground and sold it is no long vulnerable to risk. Proposition 8 abstracts from this type of prudence, since it considers the case of quadratic extraction costs (i.e.,  $G'' = 0$ ). However, within our framework where oil revenues need to be invested somewhere, oil wealth is still subject to risk after it has been

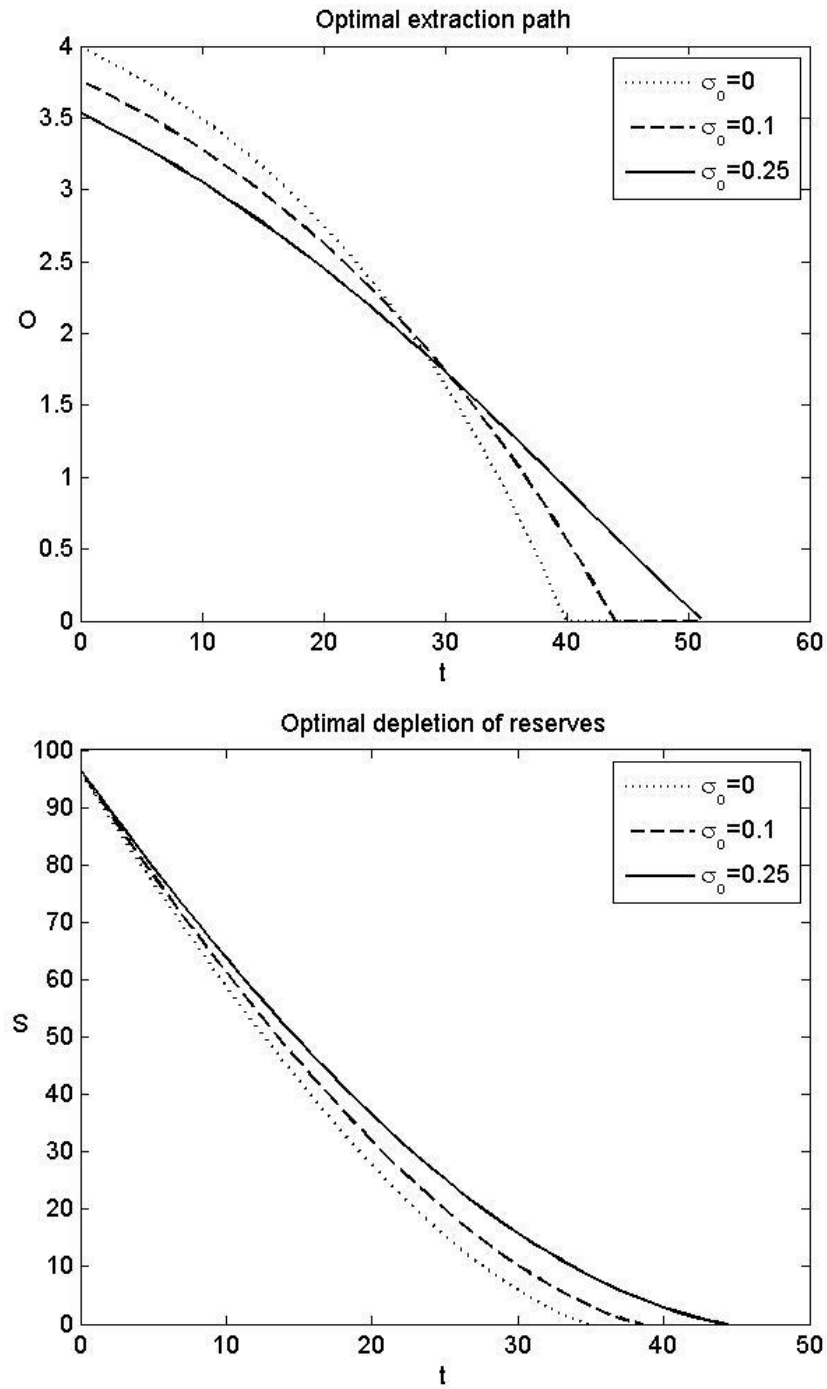
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<sup>18</sup> There is some empirical support for this (Ewing and Thompson, 2007).

<sup>19</sup> Normally, prudence relies on the convexity of marginal utility,  $U''' > 0$  (Kimball, 1990).

extracted. Extracting oil does not remove risk, but does change it. Thus the interaction between oil prices and the rest of the market is crucial.

**Figure 5: Oil extraction paths when asset A perfectly correlated with oil and asset B not all**



In practice, fluctuations in the oil price make reserves more or less economic. An example is the Athabasca oil sand reserves in Alberta, Canada. These reserves have been known since at least the 18<sup>th</sup> century. Production began very slowly in 1967 with 30,000 barrels per day. The rapid rise in oil prices since 2003 has made these reserves economically viable, and production increased to over 1.7 million barrels per day in 2011. During this period both oil prices and asset returns fell dramatically as a result of the financial crisis of 2008-2009, as did production (Alberta Energy Regulator, 2013). Proposition 8 suggests that the Albertan government should delay oil production by more than a simple forecast of oil prices would suggest. The reason is Alberta's Heritage Savings Trust Fund. Delaying oil on average would allow more oil to be produced during the boom years, thus making the best of both oil and financial markets in boom years and less during periods of general recession.

#### 4.2. Optimal portfolio allocation and consumption with endogenous rates of oil extraction

Finally, we will show that with complete markets oil rents can be fully hedged by the fund, regardless of the oil extraction path. This involves continuously adjusting the asset allocation so that the net exposure to risk at any point of time remains a constant share of total wealth. The oil extraction path should not affect consumption directly, only through its effect on the expected present value of oil rents.

**Proposition 9:** *With complete markets, continuous trading and the rate of oil extraction given by equation (25), oil wealth can be replicated with a bundle of value  $X(t)$ , comprising the perfectly correlated asset  $k$  and the safe asset  $n$ , and evolves according to*

$$(27) \quad dX(t) + \Omega(t)dt = [rX(t) + (\alpha_k - r)\omega_k(O, t)X(t)]dt + \omega_k(O, t)X(t)\sigma_k dZ_k(t),$$

where  $\omega_k(O, t) = N_k P_k / X$  is the continuously adjusted share of asset  $k$  in the replicating bundle.

Total fund and oil wealth evolves according to

$$(28) \quad dW = \sum_{i=1}^m (\alpha_i - r)\bar{w}_i W + (rW - C)dt + \sum_{i=1}^m \sigma_i \bar{w}_i W dZ_i,$$

where

$$(29) \quad \bar{w}_i(t) = w_i \left( \frac{F(t)}{F(t) + V(t)} \right), i \neq k, \quad \bar{w}_k(t) = w_k(t) \left( \frac{F(t)}{F(t) + V(t)} \right) + \omega_k(O, t) \left( \frac{V(t)}{F(t) + V(t)} \right).$$

**Proof:** See appendix A.4.

Oil rents no longer follow a Geometric Brownian Motion (as in section 3), but are driven by the drift  $\mu_{\Omega}(P_O, S, t)dt$ , and the volatility,  $\sigma_{\Omega}(P_O, S, t)dZ_O$  which depend on the states,  $P_O$  and  $S$ , and the optimally chosen rate of oil extraction which depends on those states too:

$$(30) \quad d\Omega = \mu_{\Omega}(P_O, S, t)dt + \sigma_{\Omega}(P_O, S, t)dZ_O.$$

The drift and volatility of oil rents can be replicated by continuously changing the composition of the bundle of the perfectly correlated asset and the safe asset. The challenge is to continuously adjust the amount of  $k$  in the bundle so that the instantaneous change in the value of oil rents,  $\sigma_{\Omega}(P_O, S, t)dZ_O$ , is matched perfectly by the instantaneous change in the bundle,  $\omega_k(O, t)X(t)\sigma_k dZ_k$ . The portfolio holding of the safe asset is then chosen so that the instantaneous drifts also match.

As before the fund should be managed to ensure that the net exposure to each financial asset is a constant share of total wealth:  $\bar{w}_i = \delta_i w_i, i = 1, \dots, m$  from proposition 4. Any exposure to asset  $k$  that is embodied in oil,  $\omega_k(O, t)$ , can be offset by the asset's weight in the fund,  $w$ , so as to ensure that the net weight in total wealth is constant as indicated by (29). By rearranging (28) the holdings of each financial asset in the fund can, as before, be split up into a leveraged component and an additional hedging component for the perfectly correlated asset  $k$ :

$$(31) \quad w_i = \bar{w}_i \left( \frac{F+V}{F} \right), i \neq k, \quad w_k = \underbrace{\bar{w}_k \left( \frac{F+V}{F} \right)}_{\text{leveraged demand}} + \underbrace{\left[ -\omega_k(O, t) \left( \frac{V}{F} \right) \right]}_{\text{speculative demand}}.$$

Combining proposition 5 and (28) we obtain (17), so that consumption is a constant fraction of total wealth. The rate of oil extraction does not affect consumption directly, but only through its effect on total wealth. Oil extraction and consumption are thus separated due to judicious management of the fund: on the one hand, the fund allows consumption to be smoothed in line with the permanent income hypothesis, and, on the other hand, the fund buffers consumption from oil price volatility by hedging it with traded financial assets. Only the residual volatility of total wealth (the part of oil wealth that cannot be diversified away) needs to be managed by additional precautionary saving.

## 5. Concluding remarks

Commodity exporters have two major types of national assets: natural resources below the ground and a sovereign wealth fund above it. Although some attempts to hedge commodity price volatility have been made, from long-term forward agreements in iron ore until 2010 to the purchase of oil options by Mexico

in 2008, there is no evidence of systematic coordination of below-ground and above-ground assets. We have therefore made a case for coordinating the management of these two types of asset by integrating the theories of portfolio allocation, precautionary saving, and optimal oil extraction under oil- and asset-price volatility. Our main findings are as follows. First, the fund should leverage its holdings of risky assets by a factor that is equal to the ratio of total of oil and fund wealth to fund wealth by going short and reducing its holdings of the safe asset. As natural resource reserves run out, this leveraging has to unwind. The fund thus becomes less risky as it matures. Second, oil price fluctuations have to be hedged by investing roughly relatively more in assets whose returns are negatively correlated with oil price fluctuations and vice versa, though the actual allocation should be determined by principal component analysis of the full covariance structures of all the shocks. These hedging demands are proportional to the ratio of natural resource wealth to fund wealth. Third, the rate of natural resource extraction is slowed down as a consequence of oil price volatility if oil prices are positively correlated with the financial markets, thus leaving a buffer of reserves to be extracted when oil prices and returns on financial assets are high. Fourth, consumption should be a constant share of the total of natural resource wealth and fund wealth. This means that the bird in hand rule used by Norway and other countries has to be modified to include total wealth, so that the fund helps to insulate consumption from oil extraction. Finally, precautionary saving should be used to manage the risk associated with the residual non-diversifiable volatility of oil prices.

Our analysis offers a first step towards an integrated approach to managing sovereign wealth funds and natural resources under uncertainty. Future work should allow for the uncertainty and costs associated with discoveries and exploration of new reserves by extending Pindyck (1978) to a setting with financial assets. This would help to understand how leveraging financial assets and hedging oil wealth in a sovereign wealth fund would affect exploration effort. It is also important to allow for the effect of oil price volatility on investment in domestic, non-traded capital and the growth and development of the economy. This involves extending the analysis of Gaudet and Khadr (1991) and Atewemba and Gaudet (2012) to allow for financial assets, and would ideally allow for capital scarcity too. In countries that have less access to international capital markets the optimal structure of domestic production may be reformed to be less vulnerable to commodity price volatility. A better modeling of the volatility of asset and oil prices may make our recommendations more compelling. Although lognormal distributions for commodity and asset prices lead to closed-form solutions, in practice prices exhibit mean reversion, large jumps and time-varying correlation. Finally, there may be limits on how much a country is able to hedge even if the risk is spread across many assets, national governments may be averse to taking large short positions to allow leveraged holdings of risky assets by borrowing, and large precautionary buffers may

be raided by political rivals in a partisan political framework. A better understanding of such constraints would lead to recommendations that are politically viable.

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## Appendix

### A.1 Complete markets and exogenous extraction (proof of Propositions 3 and 6):

Our proof extends the analysis of university endowments by Merton (1990, chapter 21) to both complete and incomplete markets. We first justify (20'). Let the oil price be instantaneously imperfectly correlated with the return on each asset,  $dZ_o dZ_i = \rho_{oi} dt$ ,  $\rho_{oi} \neq \pm 1$ ,  $\forall i = 1, \dots, m$ . The returns on these  $m + 1$  prices can be expressed as a linear combination of  $m + 1$  independent Wiener processes,  $du_j \square N(0, dt)$ ,  $j = 1, \dots, m + 1$ :

$$(A1) \quad \begin{pmatrix} dZ_o \\ dZ_1 \\ \vdots \\ dZ_m \end{pmatrix} = \begin{pmatrix} \lambda_{0o} & \lambda_{01} & \cdots & \lambda_{0m} \\ \tilde{\lambda}_{1o} & \tilde{\lambda}_{11} & \cdots & \tilde{\lambda}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\lambda}_{mo} & \tilde{\lambda}_{m1} & \cdots & \tilde{\lambda}_{mm} \end{pmatrix} \begin{pmatrix} du_o \\ du_1 \\ \vdots \\ du_m \end{pmatrix}.$$

Rearranging allows the instantaneous return on oil to be separated from the other assets:

$$(A1') \quad \begin{pmatrix} dZ_o \\ dZ_1 \\ \vdots \\ dZ_m \end{pmatrix} = \begin{pmatrix} \lambda_{0o} & \lambda_{01} & \cdots & \lambda_{0m} \\ 0 & \lambda_{11} & \cdots & \lambda_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_{m1} & \cdots & \lambda_{mm} \end{pmatrix} \begin{pmatrix} du_o \\ du_1 \\ \vdots \\ du_m \end{pmatrix}.$$



$$(A2) \quad dZ_O = \lambda_{0O} du_O + \Lambda_O du, \quad dZ = \Lambda du,$$

where  $\Lambda_O \equiv [\lambda_{01}, \dots, \lambda_{0m}]$ ,  $\Lambda \equiv [\lambda_{ij}]$ ,  $du \equiv [du_1, \dots, du_m]'$  and  $dZ \equiv [dZ_1, \dots, dZ_m]'$ . Equation (20') follows from

(A2) if the matrix  $\Lambda$  is invertible. The parameters  $\Lambda_O$ ,  $\Lambda$  and  $\lambda_{0O}$  can be estimated using principle components analysis. The instantaneous covariance between two Wiener processes can be expressed as  $\text{cov}(dZ_i, dZ_j) = \sum_{g=1}^m \lambda_{ig} \lambda_{jg} \text{var}(du_g)$ . The covariance matrix for  $dZ$  is  $\Sigma = (E_V E_A^{1/2} dt^{1/2})(E_V E_A^{1/2} dt^{1/2})'$ ,

where  $E_V$  is the matrix of eigenvectors,  $[E_A^{1/2}]_{ij} = e_i^{1/2}$  is the diagonal matrix of the square roots of eigenvalues,  $e_i$ , and  $\Lambda = E_V E_A^{1/2}$ . If markets are complete then  $\lambda_{0O} = 0$  and the spanned oil price can be expressed as follows:

$$(A3) \quad P_O(t) = P_O(0) \exp(-\phi t) \prod_{i=1}^m \left[ \frac{P_i(t)}{P_i(0)} \right]^{\beta_i},$$

where  $\beta_i \equiv \sigma_O M_i / \sigma_i$ ,  $M_i \equiv [\Lambda_O \Lambda^{-1}]_i$  and  $\phi \equiv -\alpha_0 + \sum_{i=1}^m \beta_i \left( \alpha_i - \frac{1}{2} \sigma_i^2 \right) + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \beta_i \beta_j \sigma_{ij}$  (from applying

Ito's lemma and comparing coefficients with equation (11)). An estimate of the covariance matrix can thus be found after estimating the parameters  $\sigma_i$ ,  $i = 1, \dots, m$  and  $\sigma_O$  by maximum likelihood.

**Lemma A1:** If markets are complete, the capitalized value of oil income (“oil wealth”) is

$$(A4) \quad V(P_O, t) = P_O(t) O \left\{ 1 - \exp[-\psi(T-t)] \right\} / \psi, \quad \psi \equiv r - \alpha_0 + \sum_{i=1}^m \beta_i (\alpha_i - r).$$

**Proof:** First, we construct a portfolio that is identical to the capitalized value of oil. Second, we construct another portfolio consisting of the risky and safe financial assets and oil wealth. Third, we show that the posited expression for oil wealth satisfies an arbitrage condition that has to hold.

First, we construct a portfolio with value  $V(P_1, \dots, P_m, t)$  which consists of assets 1, ...,  $n$  and distributes an amount of cash equal to  $P_O(t)O$  per unit time. This value evolves according to

$$(A5) \quad dV = (\mu_V V - P_O O) dt + \sigma_O V dZ_O.$$

With the aid of Ito's lemma the dynamics of the portfolio can be written as

$$(A6) \quad dV = \sum_{i=1}^m V_i P_i + V_t dt + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m V_{ij} dP_i dP_j = \left[ \sum_{i=1}^m \alpha_i V_i P_i + V_t + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m V_{ij} \sigma_i \sigma_j \rho_{ij} P_i P_j \right] dt + \sum_{i=1}^m \sigma_i P_i V_i dZ_i,$$

where subscripts of  $V(P_1, \dots, P_m, t)$  denote partial derivatives and  $\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$ . Comparing coefficients with (A5) gives

$$(A7a) \quad \mu_V V - P_O O = \sum_{i=1}^m \alpha_i P_i V_i + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \sigma_i \sigma_j \rho_{ij} V_{ij} P_i P_j,$$

$$(A7b) \quad \sigma_V V dZ_V = \sum_{i=1}^m \sigma_i P_i V_i dZ_i.$$

Finally, let  $dZ_V = \Lambda_V du$ . This implies

$$(A8) \quad \sigma_V dZ_V = \sigma_V \Lambda_V du = \Gamma' \Lambda du, \quad \Gamma \equiv [V_1 \sigma_1 P_1 / V, \dots, V_m \sigma_m P_m / V].$$

Second, we create another portfolio with value  $X(t)$  that consists of oil wealth  $V(t)$ , the risky assets and the safe asset. This portfolio is dynamically constructed, so that short positions offset the long positions, there is no net risk, and the net value of the asset is always equal to zero. Hence, the weight of the safe asset in total wealth is  $w_r = -w_V - \sum_{i=1}^m w_i$ , where  $w_V$  is the weight of oil wealth in total wealth. The return to this portfolio is

$$(A9) \quad \begin{aligned} dX &= w_V \left( \frac{dV + P_O O dt}{V} \right) + \sum_{i=1}^m w_i \left( \frac{P_i}{P} \right) + w_r r dt = \\ &\left[ w_V (\mu_V - r) + \sum_{i=1}^m w_i (\alpha_i - r) \right] dt + w_V \sigma_V dZ_V + \sum_{i=1}^m w_i \sigma_i dZ_i = \\ &\left[ w_V (\mu_V - r) + \sum_{i=1}^m w_i (\alpha_i - r) \right] dt + w_V \Gamma' \Lambda du + \Psi \Lambda du, \end{aligned}$$

where the second equality follows from (A5), the third equality from (A8) and  $\Psi \equiv [w_1 \sigma_1, \dots, w_m \sigma_m]'$ .

Suppose that the weights in this new portfolio are dynamically constructed so that there is no risk:  $w_V \Gamma' \Lambda du + \Psi \Lambda du = 0$  and the last two terms in the last equality of (A9) vanish. The weights that would achieve this are  $w_i = -(V_i / V) P_i w_V, i = 1, \dots, m$ . Arbitrage dictates that such a constructed portfolio must have a zero expected excess return over the risk-free rate:

$$(A10) \quad w_V (\mu_V - r) + \sum_{i=1}^m w_i (\alpha_i - r) = 0, \quad V (\mu_V - r) = \sum_{i=1}^m V_i P_i (\alpha_i - r).$$

Combining (A10) with (A7a) gives the following optimality condition for the portfolio:

$$(A11) \quad \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \sigma_i \sigma_j \rho_{ij} P_i P_j V_{ij} + \sum_{i=1}^m r P_i V_i - rV + V_t + P_o O = 0.$$

Third, we note that the proposed capitalized value of oil income and associated partials,

$$(A12) \quad V(P_o, t) = \frac{1}{\psi} P_o(0) O \exp(-\phi t) \prod_{i=1}^m \left[ \frac{P_i(t)}{P_i(0)} \right]^{\beta_i} \{1 - \exp[-\psi(T-t)]\},$$

$$V_i = \frac{\beta_i V}{P_i}, \quad V_t = -\phi V - P_o(t) O \exp[-\psi(T-t)],$$

$$V_{ii} = \frac{\beta_i(\beta_i - 1)V}{P_i^2}, \quad V_{ij} = \frac{\beta_i \beta_j V}{P_i P_j}, \quad j = 1, \dots, m, \quad i = 1, \dots, m.$$

indeed satisfies (A11). Lemma A1 gives the capitalized value of oil income. The instantaneous rate of change of this value is found by applying Ito's lemma to (A4):

$$(A13) \quad P_o(t) O dt + dV = \left[ r + \sum_{i=1}^m \beta_i (\alpha_i - r) \right] V dt + \sigma_o V dZ_o.$$

The results in the second part of propositions 3 and 6 follow from substituting (A13) and the evolution of fund assets given by (2') into the expression for total wealth,  $dW = dF + dV$ :

$$(A14) \quad dW = \sum_{i=1}^m (\alpha_i - r)(w_i F + \beta_i V) + (rW - C) dt + \sum_{i=1}^m w_i F \sigma_i dZ_i + \sigma_o V dZ_o.$$

## A.2 Incomplete markets and exogenous extraction (proof of proposition 7):

Now let markets be incomplete, so that the shocks underlying the oil price are not spanned by other assets in the market,

$$(A15) \quad \begin{pmatrix} dZ_o \\ dZ \end{pmatrix} = \begin{pmatrix} I_{o,0} & L_o \\ [0]_{m \times 1} & L \end{pmatrix} \begin{pmatrix} du_o \\ du \end{pmatrix}$$

with  $dZ_o$  the 1x1 Wiener process driving the oil price (both the spanned and the unspanned part),  $dZ$  the  $m \times 1$  vector of Wiener processes describing the individual above-ground assets,  $du_o$  the 1x1 Wiener processes that drives oil price fluctuations that cannot be spanned by the market and  $du$  the  $m \times 1$  vector of fundamental Wiener processes underlying the market. The elements of the vectors  $du_o$  and  $du$  are Wiener processes with zero correlation. The vector of zeros  $[0]_{m \times 1}$  ensures that the shocks to the above-ground assets  $dZ$  are not affected by shocks in the fundamental oil price process  $du_o$ . The scalar  $I_{o,0}$

determines the share of the oil price that is spanned. Since  $du_o$  and  $dZ_o$  both have a variance of unity (cf. Wiener processes), we must have:

$$(A16) \quad \lambda_{o,0}^2 + \sum_{i=1}^m [\lambda_{o,i}]^2 = 1,$$

Total oil price volatility is thus unaffected by the degree of spanning. From (A15) we have

$$(A17) \quad dZ = \Lambda du \Rightarrow dZ_o = \lambda_{o,0} du_o + \Lambda_o du = \lambda_{o,0} du_o + \Lambda_o \Lambda^{-1} dZ.$$

The oil price in (A3) can be rewritten as,

$$(A3') \quad P_o(t) = P_o(0) \exp(-\phi t) \prod_{i=1}^m \left[ \frac{P_i(t)}{P_i(0)} \right]^{\beta_i} \left[ \frac{P_h(t)}{P_h(0)} \right]^{\beta_h},$$

where

$$(A18) \quad \beta_i = \frac{\sigma_o}{\sigma_i} [M]_i = \frac{\sigma_o}{\sigma_i} [\Lambda_o \Lambda^{-1}]_i \quad \text{for } \forall i \leq m, \quad \beta_h = \frac{\sigma_o}{\sigma_h} \lambda_{o,0}.$$

and  $P_h(t)$  is the unspanned component of oil prices, which we assume also follows the process,

$$(A19) \quad dP_h = a_h P_h dt + S_h P_h du_o.$$

Let the value function be  $J(F, V, t) = E_t \left[ \int_t^\infty U(C) e^{-rt} dt \right]$ , where  $F$  is above-ground wealth and  $V$  is

below-ground wealth. Above-ground wealth is accumulated according to

$$(A20) \quad dF = \sum_{i=1}^m w_i F (\alpha_i - r) dt + (rF + P_o O - C) dt + \sum_{i=1}^m w_i F \sigma_i dZ_i.$$

Following the derivation of (A13) we get the evolution of below ground wealth:

$$(A13') \quad dV + P_o O dt = \left( r + \sum_{i=1}^m \beta_i (\alpha_i - r) + \beta_h (\alpha_h - r) \right) V dt + \sigma_o V (\Lambda_o \Lambda^{-1} dZ + \lambda_{o,0} du_o),$$

It can be shown by differentiating using Ito's lemma that (A13') has the following solution:

$$(A22) \quad V = P_o(t) O (1 - e^{-\psi(T-t)}) / \psi = P_o(0) e^{-\phi t} \sum_{i=1}^m \left( \frac{P_i(t)}{P_i(0)} \right)^{\beta_i} \left( \frac{P_h(t)}{P_h(0)} \right)^{\beta_h} O (1 - e^{-\psi(T-t)}) / \psi,$$

$$\phi = -\alpha_o + \sum_i^m \beta_i (\alpha_i - \sigma_i^2 / 2) + \beta_h (\alpha_h - \sigma_h^2 / 2) \text{ and } \psi = r - \alpha_o + \sum_i^m \beta_i (\alpha_i - r) + \beta_h (\alpha_h - r).$$

The Hamilton-Jacobi-Bellman equation is

$$(A23) \quad \max_{w,C} \left[ U(C) e^{-\rho t} + \frac{1}{dt} E_t [dJ(V, P_o, t)] \right] = 0,$$

where we have:

$$(A24) \quad \begin{aligned} & \frac{1}{dt} E_t [dJ] = J_t + J_F \left( \sum_{i=1}^m w_i F (\alpha_i - r) + rF + P_o O - C \right) \\ & + J_V \left( \left( r + \sum_{i=1}^m \beta_i (\alpha_i - r) + \beta_h (\alpha_h - r) \right) V - P_o O \right) + \frac{1}{2} J_{FF} F^2 \sum_{i=1}^m \sum_{j=1}^m w_i w_j \sigma_{ij} \\ & + \frac{1}{2} J_{VV} V^2 \left[ \sum_{i=1}^m \sum_{j=1}^m \beta_i \beta_j \sigma_{ij} + \sigma_o^2 \lambda_{o,0}^2 \right] + J_{VF} VF \sum_{i=1}^m \sum_{j=1}^m w_i \beta_j \sigma_{ij}. \end{aligned}$$

The first-order conditions with respect to  $C$  and  $w_i$  are:

$$(A25) \quad U'(C) e^{-\rho t} - J_F = 0 \quad \text{and} \quad J_F = U'(C) e^{-\rho t},$$

$$(A26) \quad J_F F (\alpha_i - r) + J_{FF} F^2 \overset{m}{\underset{j=1}{\overset{\Delta}{\sum}}} w_j \sigma_{ij} + J_{FV} FV \overset{m}{\underset{j=1}{\overset{\Delta}{\sum}}} b_j \sigma_{ij} = 0.$$

Equation (A26) can be solved to give the optimal weights in the fund:

$$(A27) \quad w_i = -\frac{J_F}{FJ_{FF}} \sum_{j=1}^m v_{ij} (\alpha_j - r) - \frac{J_{FV}}{J_{FF}} \frac{V}{F} \beta_i \quad \text{or} \quad w_i = \frac{C/F}{\partial C / \partial F} \theta \sum_{j=1}^m v_{ij} (\alpha_j - r) - \frac{\partial C / \partial V}{\partial C / \partial F} \frac{V}{F} \beta_i,$$

To proceed analytically we can approximate the partial derivatives with those from the complete markets

case in (17),  $\frac{\partial C}{\partial F} = \frac{\partial C}{\partial V} = \frac{\partial C}{\partial W} = r_w^*$  (or alternatively we assume that consumption is a linear function of

the sum of above- and below-ground wealth), to obtain:

$$(A28) \quad w_i \approx \frac{W}{F} \theta \sum_{j=1}^m v_{ij} (\alpha_j - r) - \frac{V}{F} \beta_i.$$

Alternatively,  $w_i = \frac{W}{F} \bar{w}_i - \frac{V}{F} \beta_i$ , where  $\bar{w}_i$  is the weight of each asset in total wealth, defined as in proposition 1 by  $w_i = w \delta_i$ , for all the assets  $i$  that span oil. The Euler equation can be written using the same approximation as

$$(A29) \quad \frac{1}{dt} E_t[dC] \approx \theta(r - \rho)C + \frac{1}{2} CRP \frac{\frac{1}{dt} E_t[dC^2]}{C^2}.$$

Note that we have:

$$(A30) \quad \frac{1}{dt} E_t[(dC)^2] = C_F^2 \frac{1}{dt} E_t[(dF)^2] + C_V^2 \frac{1}{dt} E_t[(dV)^2] + 2C_V C_F \frac{1}{dt} E_t[dVdF],$$

$$(A31) \quad \frac{1}{dt} E_t[(dF)^2] = F^2 \sum_{i=1}^m \sum_{j=1}^m w_i w_j \sigma_{ij},$$

$$(A32) \quad \frac{1}{dt} E_t[(dV)^2] = V^2 \left( \sum_{i=1}^m \sum_{j=1}^m b_i b_j S_{ij} + S_{O,0}^2 / \sigma_O^2 \right),$$

$$(A33) \quad \frac{1}{dt} E_t[dVdF] = VF \sum_{i=1}^m \sum_{j=1}^m \bar{a}_i \bar{a}_j w_i b_j S_{ij},$$

Combining these expressions we get:

$$(A34) \quad \begin{aligned} \frac{1}{dt} E_t[dC^2] &= C_W^2 \left[ \sum_{i=1}^m \sum_{j=1}^m (w_i F + \beta_i V)(w_j F + \beta_j V) \sigma_{ij} + \lambda_{O,0}^2 \sigma_O^2 V^2 \right] \\ &= C_W^2 \left( \sum_{i=1}^m \sum_{j=1}^m \bar{w}_i \bar{w}_j \sigma_{ij} W^2 + \lambda_{O,0}^2 \sigma_O^2 V^2 \right) \\ &= C_W^2 \left( \bar{w}^2 \sigma_w^2 W^2 + \lambda_{O,0}^2 \sigma_O^2 V^2 \right). \end{aligned}$$

where the weight and volatility of risky assets in total wealth,  $w^2$  and  $\sigma^2$ , are defined as in proposition 1. Hence, the stochastic Euler equation can approximately be written as equation (21) in proposition 7.

### A.3 Complete markets and endogenous extraction (proof of proposition 8):

The Hamilton-Jacobi-Bellman equation for the problem (1''), (2'') and (22) is

$$0 = \max_{C, w_i, O} \left\{ U(C)e^{-\rho t} + \frac{1}{dt} E_t [dJ(F, P_O, S, t)] \right\},$$

$$(A35) \quad \frac{1}{dt} E_t [dJ(F, P_O, S, t)] = J_F \left[ \sum_{i=1}^m w_i (\alpha_i - r) F + rF - C + P_O O - G(O) \right] J_P \alpha_O P_O - J_S O + J_t +$$

$$\frac{1}{2} J_{FF} F^2 \sum_{i=1}^m \sum_{j=1}^m w_i w_j \sigma_{ij} + \frac{1}{2} J_{PP} \sigma_O^2 P_O^2 + J_{FP} \sigma_O P_O F \sum_{i=1}^m w_i \sigma_i \rho_{iO}.$$

The first-order conditions are

$$(A36a) \quad U'(C)e^{-\rho t} = J_F,$$

$$(A36b) \quad J_F F (\alpha_i - r) + J_{FF} F^2 \sum_{i=1}^m w_j \sigma_{ij} + J_{FP} F \sigma_i \sigma_O \rho_{iO} P_O = 0,$$

$$(A36c) \quad J_F [P_O - G'(O)] - J_S = 0.$$

Upon differentiation of (A35) with respect to the state variables, we get:

$$(A37a) \quad \frac{1}{dt} E [dJ_F] + J_F \left[ r + \sum_{i=1}^m w_i (\alpha_i - r) \right] + J_{FF} F \sum_{i=1}^m \sum_{j=1}^m w_i w_j \sigma_{ij} + J_{FP} \sigma_O P_O \sum_{i=1}^m w_i \sigma_i \rho_{iO} = 0,$$

$$(A37b) \quad \frac{1}{dt} E [dJ_S] = 0,$$

$$(A37c) \quad \frac{1}{dt} E [dJ_P] + J_F O + J_P \alpha_O + J_{PP} \sigma_O^2 P_O + J_{FP} F \sum_{i=1}^m w_i \sigma_i \sigma_O \rho_{iO} = 0.$$

Upon substitution of (A36b) into (A37a), we get

$$(A38) \quad \frac{1}{dt} E [dJ_F] = -r J_F.$$

Equation (A37b) states that oil is extracted so that the marginal utility of an extra barrel of oil in the ground is always constant. Equation (A38) requires that the marginal utility of assets (or of consumption from (A36a)) must fall at the rate of interest. Combining (A36a) and (A38) gives the Euler equation

$$(A39) \quad \frac{\frac{1}{dt} E_t [dC]}{C} = \left[ \frac{-U'(C)}{CU''(C)} \right] (r - \rho) - \left[ \frac{CU'''(C)}{U''(C)} \right] \frac{\frac{1}{dt} E_t [dC^2]}{C^2}.$$

Applying Ito's lemma to (A36c) gives rise to

$$(A40) \quad \frac{dJ_S}{J_S} = \frac{dJ_F}{J_F} + \frac{d\Omega_O}{\Omega_O} + \frac{dJ_F d\Omega_O}{J_F \Omega_O}, \quad \Omega_O = P_O - G'(O).$$

Combining (A37b), (A38) and (A40) yields the expected Hotelling rule (23) of proposition 8. The extraction path is thus affected by the marginal utility of wealth and marginal oil rents. Marginal oil rents are a function of the oil price and marginal extraction cost. Both the marginal utility of wealth and the rate of oil extraction will be a function of the four state variables,  $J_F(F, P_O, S, t)$  and  $O(F, P_O, S, t)$ . Application of Ito's lemma to both yields:

$$(A41) \quad dJ_F = -rJ_F dt + J_{FF}F \sum_{i=1}^m w_i \sigma_i dZ_i + J_{FP}P \sigma_O dZ_O,$$

$$(A42) \quad dO = \mu_O(F, P_O, S, t)dt + O_F F \sum_{i=1}^m w_i \sigma_i dZ_i + O_P P \sigma_O dZ_O,$$

where we have used (A38) and  $\mu_O(F, P_O, S, t) = \frac{1}{dt} E[dO]$  is the yet to be determined expected rate of oil extraction. Applying Ito's lemma to  $\Omega_O = P_O - G'(O) = P_O - \gamma O$  gives

$$(A43) \quad \begin{aligned} d\Omega_O &= dP_O - G''(O)dO - \frac{1}{2}G'''(O)dO^2 = \\ &[\alpha_O P_O - \gamma \mu_O(F, P_O, S, t)]dt - \gamma O_F F \sum_{i=1}^m w_i \sigma_i dZ_i + (1 - \gamma O_P)P_O \sigma_O dZ_O. \end{aligned}$$

Multiplying (A41) and (A43) gives

$$(A44) \quad \begin{aligned} \frac{dJ_F d\Omega_O}{J_F \Omega_O} &= -\frac{\gamma O_F F}{J_F \Omega_O} \left\{ \sum_{i=1}^m w_i \sigma_i \left[ J_{FP} \sigma_O P_O \rho_{iO} + J_{FF} F \sum_{j=1}^m w_j \sigma_j \rho_{ij} \right] \right\} dt \\ &+ \frac{(1 - \gamma O_P) P_O \sigma_O}{J_F \Omega_O} \left( J_{FP} P_O \sigma_O + J_{FF} F \sum_{i=1}^m w_i \sigma_i \rho_{iO} \right) dt. \end{aligned}$$

This can be simplified by substituting in the optimal asset weight condition in (A36b) for all assets (the  $O_F$  term) and for the perfectly correlated asset  $k$  (the  $O_P$  term) to give

$$(A45) \quad \frac{dJ_F d\Omega_O}{J_F \Omega_O} = \frac{1}{\Omega_O} \left[ \gamma O_F F \sum_{i=1}^m w_i (\alpha_i - r) - (1 - \gamma O_P) \left( \frac{\alpha_k - r}{\sigma_k} \right) P_O \sigma_O \right] dt.$$

The expected Hotelling rule (23) together with equations (A43) and (A45) yield equation (24) of proposition 8. To get an approximation to the optimal oil extraction path we approximate the partial derivatives  $O_F$  and  $O_P$  by taking them from the deterministic solution.



**Lemma A2:** If all prices are deterministic,  $\dot{\Omega}_O = r\Omega_O$ ,  $O(t) = \frac{1}{\gamma} P_O(t) [1 - e^{-(r-\alpha_O)(T-t)}]$  and the date of exhaustion is  $T = T(S(0), P(0), \alpha_O, r, \gamma)$ .

**Proof:** In the deterministic case the extraction problem is separated from the portfolio allocation problem and oil prices follow the trend path  $\dot{P}_O = \alpha_O P_O$ ,  $\alpha_O < r$ . Given the Hotelling rule  $\dot{\Omega}_O = r\Omega_O$ , we have

$\dot{O}(t) = rO(t) - \frac{1}{\gamma} P_O(0)(r - \alpha_O)e^{\alpha_O t}$ . The solution to this differential equation with the terminal condition

$O(T) = 0$  is  $O(t) = \frac{1}{\gamma} P_O(t) [1 - e^{-(r-\alpha_O)(T-t)}]$ . Integrating this solution gives

$\int_0^T O(t) dt = \frac{1}{\gamma} P_O(0) \left[ \frac{e^{\alpha_O T} - 1}{\alpha_O} - \frac{e^{\alpha_O T} - e^{-(r-\alpha_O)T}}{r} \right] = S_0$ . This condition implicitly gives the date of

exhaustion and the comparative statics follow from total differentiation of this condition.  $\square$

Note that if oil is abundant oil supply has a price elasticity of one,  $O(t) = P(t) / \gamma$ . In general, oil is scarce and implies that supply is less elastic. Lemma A2 gives  $O_F \cong 0$  and  $O_P \cong [1 - e^{-(r-\alpha_O)(T-t)}] / \gamma$ . The approximate stochastic Hotelling rule comes from substituting these partial derivatives into (24):

$$(A46) \quad d\Omega_O \cong r\Omega_O dt - \frac{\sigma_O}{\sigma_k} P_O e^{-(r-\alpha_O)(T-t)} [(\alpha_k - r)dt - \sigma_k dZ_O].$$

The approximate expected oil extraction path comes from combining the expressions for  $\frac{1}{dt} E[d\Omega_O]$  in

equations (A46) and (A43) giving  $\mu_O(F, P_O, S, t) = \frac{1}{dt} E[dO]$  and

$$(A47) \quad \frac{1}{dt} E[dO(t)] = rO(t) - \frac{1}{\gamma} P_O(t) \left[ r - \alpha_O - \frac{\sigma_O}{\sigma_k} (\alpha_k - r) e^{-(r-\alpha_O)(T-t)} \right].$$

The path for  $dO$  then follows from substituting this into equation (A42). To find the approximate extraction path (26) in proposition 8 we treat the expected dynamics as deterministic. The solution of (A47) is then given by

$$(A48) \quad O(t) \cong \frac{1}{\gamma} P_O(t) \left[ 1 - e^{-(r-\alpha_O)(T-t)} - \frac{\sigma_O}{\sigma_k} (\alpha_k - r) e^{-(r-\alpha_O)(T-t)} (T-t) \right].$$

The expected date of exhaustion can be solved for from the condition

$$(A49) \quad \int_0^T O(t)dt = \frac{P_o(0)}{\gamma\alpha}(e^{\alpha_o T} - 1) - \left[ \frac{P_o(0)}{\gamma r} e^{-(r-\alpha_o)T} + \frac{Y}{r^2} \right] (e^{rT} - 1) + Y \frac{T}{r} = S(0),$$

where  $Y \equiv P_o(0)(\alpha_k - r)e^{-(r-\alpha_o)(T-t)} \sigma_o / \gamma \sigma_k$ .

#### A.4 Complete markets and endogenous extraction (proof of proposition 9):

Let there be a traded asset  $k$ , which is perfectly correlated with oil,  $dZ_o = dZ_k$ . The first part of the proposition states that oil rents can be replicated with a bundle containing  $N_k$  shares of asset  $k$  and  $N_r$  shares of the safe asset,  $X \equiv N_k P_k + N_r P_r$ . This bundle must yield a continuous dividend exactly equal to the optimal oil rents  $\Omega$ . This replicating bundle can be constructed as follows.

We begin in discrete time with sample period  $h$  before moving to continuous time by  $h \rightarrow 0$  following Merton (1990, p. 125). Construct a bundle so that at every time  $t$  the number of shares  $N_k(t)$  and  $N_r(t)$  are chosen and held until time  $t + h$ . At the same time a dividend is declared exactly equal to  $\Omega(t)$ , which is paid continuously throughout the period  $h$ . The bundle will start period  $t$  with  $N_k(t-h) P_k(t) + N_r(t-h) P_r(t)$  as the number of shares in each asset has been chosen in the previous period. At time  $t$  the dividend and new shares are chosen to preserve the value of the bundle,  $-\Omega(t)h = \sum_{i=k,r} [N_i(t) - N_i(t-h)] P_i(t)$ . The same must be true at  $t + h$ , so that

$$(A50) \quad \begin{aligned} -\Omega(t+h)h &= \sum_{i=k,r} [N_i(t+h) - N_i(t)] P_i(t+h) = \\ &= \sum_{i=k,r} [N_i(t+h) - N_i(t)] [P_i(t+h) - P_i(t)] + \sum_{i=k,r} [N_i(t+h) - N_i(t)] P_i(t). \end{aligned}$$

Taking the limit as  $h \rightarrow 0$  we get  $-\Omega dt = \sum_{i=k,r} dN_i dP_i + dN_i P_i$  assuming all variables are continuous. This

describes the rate at which shares have to be sold to finance the dividend.

Equation (27) of proposition 9 combines this expression for the dividends with the path for the replicating bundle. By Ito's lemma the replicating bundle must satisfy

$$(A51) \quad \begin{aligned} dX + \Omega dt &= \sum_{i=k,r} (N_i dP_i + dN_i P_i + dN_i P_i) + \Omega dt = \sum_{i=k,r} (N_i dP_i) = \\ &= \omega_k X (\alpha_k - r) dt + rX dt + \omega_k X \sigma_k dZ_k. \end{aligned}$$

where  $\omega_k \equiv N_k(t)P_k(t) / X(t)$  is the weight of the risky asset  $k$  in the replicating bundle. The weights  $\omega_k(t)$  must be updated continuously to match the stochastic path of oil rents described by (A50). We have

focused on an expression for  $dV(t) + \Omega(t)dt$ . The explicit expression for  $V(t)$  can be found using contingent claims analysis (Merton, 1990). This is also applicable when oil rents follow the general Ito process  $d\Omega(t) = a(\cdot)\Omega dt + s(\cdot)\Omega dZ_o$  when  $a(\cdot)$  and  $s(\cdot)$  are not constants. The value of oil rents must equal that of the replicating bundle,  $V(t) = X(t)$ , because both share exactly the same properties (made possible by the perfect correlation between asset  $k$  and oil). Equation (27) of proposition 9 states that the policy maker's problem can be summarized in terms of total wealth. Total wealth is given by  $W(t) = F(t) + V(t)$ . Combining equations (2'') and (A51) gives equation (28) in proposition 9